Is Life Computable?

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Abstract. This paper has two primary aims. The first is to provide an introductory discussion of hyperset theory and its usefulness for modeling complex systems. The second aim is to provide a hyperset analysis of Robert Rosen’s metabolism-repair systems and his claim that living things are closed to efficient cause. Consequences of the hyperset models for Rosen’s claims concerning computability and life are discussed.

1. Introduction

Toward the end of his book Life Itself, Robert Rosen (1991; see also Rosen 2000) claimed that the defining feature of living systems is that they are “closed to efficient cause”. The idea of closure to efficient cause is wrapped up with a series of other concepts (metabolism-repair systems, computability, impredicativity, complexity) in a somewhat obscure way. (See Letelier, Soto-Andrade, Abarzúa, Cornish-Bowden and Cárdenas 2006 for a recent attempt at clarification.) One aim of this paper is to set out exactly how these concepts are related. Rosen’s view was that closure to efficient causation is a variety of complexity, where complex systems are systems whose models contain impredicativities, and, therefore, are not computable. We will see in this paper that the relationship between complexity and computability is not exactly the one that Rosen imagined, but that his overall approach is still vindicated. In particular, we will use hyperset theory to find fault with Rosen’s claim that, because of this complexity of living systems, the mathematical and computational tools of the mainstream cognitive sciences (i.e., computability theory and computational models) are inappropriate as tools for analyzing systems that are closed to efficient causation (e.g., metabolism-repair systems). So, a further, and more important aim of this paper, is to introduce non-well founded set theory or hyperset theory (Aczel 1988, Barwise and Etchemendy 1987, Barwise and Moss 1996, Kercel 2003), and show that it is a useful tool to make sense of the kind of complexity that is characteristic of living systems.

Our plan is as follows. We will begin by introducing non-well founded set theory or hyperset theory. Hyperset theory comes with an intuitive, easy-to-understand system of graphs that make it possible to test systems for complexity. In the next section, we will discuss hypersets and introduce the graphing system. We will show that complex systems will all have graphs with a similar structure—they all have loops. In the following section, we will use these tools to analyze Robert Rosen’s...
metabolism-repair systems (1991). The purpose of this will be to show that, as suggested by Kercel (2003), the diagrams do indeed capture the complexity of Rosen’s model. Furthermore, it will allow us to say what distinguishes systems that are closed to efficient cause from other complex systems. In the last section, we argue that the hyperset graphs suggest that Rosen was incorrect about the non-computability of living systems.

2. Introducing Hyperset Theory

During the late 19th and early 20th century work that codified and established set theory, it became evident that certain paradoxes threatened the foundations of set theory. The most famous of these paradoxes was first discussed by Bertrand Russell, and is now known as Russell’s Paradox (1903). Imagine the barber who shaves all and only those who don’t shave themselves. Who shaves the barber? It turns out that if the barber shaves himself, he does not shave himself, and vice versa. Put in terms of sets, the paradox is as follows. Call all the sets that do not contain themselves as members normal sets; those that do contain themselves are abnormal sets. Consider S the set of all normal sets. Is S normal? If so, S is a member of the set of all normal sets. But since S is the set of all normal sets, S would then be abnormal. If S is abnormal, then S is a member of itself. But if S is a member of itself (the set of all normal sets), then S must be normal. So: S is a member of itself just in case it is not a member of itself, or, equivalently, S is normal just in case it is abnormal. This means that the sentence “S is a normal set” is simultaneously true and false.

This was taken to be a serious problem with the theory of sets. In response to this problem, Russell introduced the vicious circle principle, which outlawed any sentence which was circular. This was later formalized as the theory of types, most of the details of which we do not need to be concerned with, but whose main effect was to make it illegal for sets to be members of themselves. According to the theory of types, that is, there are no abnormal sets, and there is also no set of all sets. Paradox solved. Poincaré’s (1906) banishment of impredicative definitions from mathematics, which may be more familiar to some readers, serves a similar purpose. Roughly, a predicative definition applies to members of some domain so that its application is not altered by addition of new members to the domain; an impredicative definition, on the other hand, picks out different members of a domain should new individuals be added to the domain. Essentially, impredicative definitions pick out individuals or properties whose falling under that definition depend on other members of a set. Another way to put this is that impredicative definitions pick out individuals in a way that is context-dependent. “Abnormal set” and “the barber who shaves all and only those who do not shave themselves” are defined impredicatively, so outlawing impredicativities makes defining these types of set impossible. Again, paradox solved.

The elimination of abnormal sets leaves us with standard set theory as taught in College-level logic and mathematics courses. This set theory is well-founded. There are two related senses in which standard set theory is well-founded. First, it is well-founded in that its implementation does not lead to paradoxes; that is, it is logically well-founded. The second way in which it is well-founded comes from graph theory,
a standard tool used to understand set membership. In graph theory, every set determines a tree-like graph in which the set itself is the upper-most node and each member of the set is a descendent node with an arrow pointing to it from the original set of which it is a member; this is applied recursively. Figure 1 shows a graph for the set \( P = \{ a, b, c \} \). Notice that there is an arrow from the node representing the set \( P \) to each of its members. Figure 2 shows a graph for the set \( Q = \{ a, b, \{ c \} \} \). Notice that in this graph, there is an additional complexity in which the node for the member of \( Q, \{ c \}, \) has an additional member, the object \( c \). The sets \( P \) and \( Q \) are well-founded sets; they are not abnormal and do not contain themselves as members. The graphs of \( P \) and \( Q \) show a property common to the graphs of all well-founded sets. Graphs of well-founded sets contain no infinite cycles or loops, and each path from the original set terminates in some object (its foundation, as it were). In fact, Aczel 1988 proves that a graph will contain no cycles or loops if and only if it is well founded. The ‘if and only if’ means that a graph containing loops or cycles is a picture of a non-well-founded set. The presence of cycles and/or loops, that is, would indicate that some set has itself as a member, or that the concept, system, or definition it models is impredicative.

Consider, as an example, the abnormal set \( A = \{ A, b, c \} \). This will have the graph in Figure 3. Aczel also defines the special abnormal set \( W \) as the set that contains only itself, \( W = \{ W \} \). \( W \) is graphed in Figure 4. The sets \( A \) and \( W \) are
not allowed in well-founded set theory, but they do have coherent and useful graphs. Aczel’s Anti-Foundation Axiom that embraces non-well-founded (i.e., abnormal) sets is as follows: every graph, with or without loops, pictures a genuine set. Hypersets are defined as graphable sets. So well founded sets and non-well-founded sets are both types of hyperset.

Non-well-founded sets were banished from set theory in the interest of the logical soundness of the theory, and with the explicit goal of reducing mathematics to set theory, and set theory to logic. Re-introducing non-well-founded sets into our set theory makes such reduction impossible. But this loss (if it really is a loss) is offset by the increased ability to model real-world phenomena. Work by Gupta (1981), Barwise and Etchemendy (1987), Gupta and Belnap (1993) and Barwise and Moss (1996) makes very clear that many concepts and real-world systems are circular or otherwise not-well-founded, hence illegal according to standard set theory. The most well known example is the concept ‘truth’, and the predicate ‘is true’. Gupta (1981) argues that the semantic paradoxes involving truth, such as the liar (“This very sentence is false”) and the truth-teller (“This very sentence is true”) show that the predicate “is true” can only be defined circularly, and develops a logical technique (the revision theory) for dealing with circular concepts. Later, Barwise and Etchemendy (1987) use hypersets to understand the truth predicate.

In each of these cases, some object in a system is defined impredicatively, and is therefore illegal in standard set theory. Of course, the illegality of these systems and concepts in standard set theory does not change their behavior, or cause them to cease to exist. Such systems do exist. Adequately modeling them requires a richer set theory, such as hyperset theory, that allows and accounts for the behavior of systems with impredicativities.

The graph theory that goes with hyperset theory will be especially important in what follows. As Aczel has proved and figures 1-4 demonstrate, graphs of hypersets can be used to determine whether the set pictured is or is not well-founded, and if it is not well-founded exactly what it is that makes it non-well-founded. These graphs, that is, can be used to determine whether systems contain impredicativities, and so are complex. The graphs also allow us to determine the precise source and nature of the complexity in such systems. We will use this feature of hypersets and their graphs to look at Rosen’s model of living things as metabolism-repair systems.

3. Rosen on Life Itself

According to Rosen (1991, 2000), a central feature of living things is complexity. Rosen, it should be noted, has an idiosyncratic understanding of complexity. First, according to Rosen, whether a system is complex depends only secondarily on the system itself; it depends primarily on the system’s models. A physical system is complex if and only if it has impredicative models; otherwise, the system is simple. (As mentioned in Section 1 above, Rosen typically described this in terms of Turing-computability. We discuss this issue below.) This connects with the above discussion of impredicative definitions and Russell’s theory of types. Put most simply, Russell introduced the theory of types and Poincaré outlawed impredicative
definitions specifically to keep logic, set theory and mathematics well-founded. Thus systems that have models that disobey the theory of types or contain impredicativities — models, that is, that are not well-founded — are complex in Rosen’s sense. If this is so, we can use hyperset diagrams as a means of modeling systems that are complex. Indeed, we can use hyperset diagrams as complexity detectors.

Fig. 5. Rosen's model of living systems (1991). Note that \( Y = H(X) \) is represented by a solid line from \( H \) to \( X \) and a dashed line from \( X \) to \( Y \).

We can see this by examining Rosen’s metabolism-repair model of living systems (1991, 2000). According to Rosen, it is a necessary condition on living systems that their models contain three functions: a metabolic function, a repair function, and a replication function. We will discuss these in order. The metabolic function \( h_i \) is a mapping from raw materials \( x_i \in X \) to behavior \( y_i \in Y \). That is, it determines the way the system reacts to environmental input. A living system must also have a repair function that repairs the metabolic function to keep the system reacting appropriately to the environment. That is, it must have a function \( a \) which maps behaviors \( y_i \in Y \) onto the members of set \( H \), where metabolic \( h_i \in H \). Finally, the repair function must itself be repaired. This repair is done by the system’s behavior. That is, each member \( y_i \) of the set of behaviors \( Y \) is itself a replication function that maintains the repair function, by mapping \( h_i \) onto \( a_i \in A \). This is, admittedly, complicated. The idea, though, is rather simple. The living system must be metabolizing consistently, so must maintain the causal processes that it uses in metabolizing. This is the job of the repair function. But the repair function must also be maintained, and this is done by the system’s behavioral output. Rosen’s diagrammatic representation of his model of living systems is shown in Figure 5. Note that in this diagram a function \( P \) from \( Q \) to \( R \) is represented as ‘\( P \)’ connected to ‘\( Q \)’ with a solid arrow, and ‘\( Q \)’ connected to ‘\( R \)’ with a dashed arrow.

Figure 6 is a hyperset diagram of Rosen’s model of living systems. In figure 6, two common set-theoretic conventions are followed. First, functions are represented as ordered pairs. That is, if \( P \) is a function from \( Q \) to \( R \), \( P \) is represented as \( P = \langle Q, R \rangle \). Second, to preserve their ordering, ordered pairs are represented as sets containing the set containing their first member and the set containing both their members. Thus \( \langle Q, R \rangle \) is represented as \( \{ \{ Q \}, \{ Q, R \} \} \). In Figure 6, the sets of inputs \( X \),
behaviors Y, metabolic functions H, and repair functions A are represented as if they were individuals. This is done to make the already-complicated diagram simpler.

**Fig. 6.** Hyperset diagram of Rosen’s metabolism-repair system. Functions are represented as ordered pairs containing their domain and range. So f(a) = b is represented as f = <a, b>.

The thing to notice about the hyperset diagram in figure 6 is that there are loops, indicating that the sets are not well-founded. Indeed, in the graph there is only one location where the system does not loop back on itself: the set X of inputs from the environment. This makes sense of two important aspects of Rosen’s model of life. First, the loops indicate that Rosen’s diagram is indeed complex; it involves irremovable impredicativities. Rosen took just this kind of complexity to be one of the most important features of living systems, and often wrote of the impredicativities found in living systems. Figure 6 bears this out. Second, the fact that the only place the diagram “bottoms out” is with raw materials is also important. Rosen often wrote that living systems are “closed to efficient causation”. In Rosen’s Aristotelian language, an ‘efficient cause’ matches up with our common sense intuition about the cause of an event. Thus, the efficient cause of a billiard ball moving is that it was struck by another billiard ball. When Rosen says that living systems are ‘closed to efficient cause’, he means that all of the efficient causes of the system are produced by the system. Entities and events outside the system cannot be efficient causes of its behavior, they can only be raw material (Aristotelian ‘material causes’) for the system. We can see this in figure 5, Rosen’s own representation of living systems. In
that diagram, only X, the raw material input from the environment, is not both (a) itself a function in the system and (b) the output of a function of the system. To put this in Rosen’s (and Aristotle’s) language, all functions but X are both efficient and material causes in the system. X is only a material cause. The hyperset diagrams make clear how this is the case.

4. Conclusions Concerning Computability and Life

The hyperset analysis of metabolism-repair systems is germane to the debate over the possibility of artificial life. Rosen’s central argument is that organisms will have impredicative models, which he takes to mean say that they have models that are not Turing computable, which is to say that they are not machines. We have demonstrated using hypersets that metabolism-repair systems have impredicative models. Following Rosen, one might conclude that these systems are not Turing-computable. Considering the nature of impredicativity (and non-well-founded sets) it is not hard to see why Rosen made this inference. A set S is Turing-computable if and only if a Turing machine given any input n from S’s domain halts with output 1 if n is in S and halts with 0 if n is not in S. Now consider the case of circularly defined sets. Suppose that S = { … A … } and A = { … S … }. A Turing machine T that can determine whether some n is a member of S will need to check the members of S to determine whether n is among them. In doing so, T will need to check the membership of each set that is a member of S. In this case, it will check the membership of A. To determine the membership of A, it will need to determine the membership of all the sets that are members of A, including S in this case. This, it seems, leads to an infinite loop, which would mean that T does not halt. So it would seem that S and A are not Turing-computable.

There is, however, no necessary connection between impredicative definitions and non-Turing-computability. For example, work by Martin-Löf (1970) and Girard (1972) on impredicative types has been extremely valuable in computer science. More importantly, hyperset membership has been shown to be Turing-computable in polynomial time (Lisitsa and Sazonov 1999; Sazonov, 2006), a result anticipated in the pioneering work of Barwise and Moss (1996). Deciding on hyperset membership therefore does not necessarily involve infinite loops with their unwelcome consequences of adding infinitely many copies of elements to sets and an inability to terminate. Thus Rosen’s claim that “there is no algorithm for building something that is impredicative” (2000, p. 294) is incorrect. The question at this point is to what extent the Turing-computability of impredicative sets, whether modeled as hypersets or not, affects Rosen’s most important contributions.

Rosen claimed that impredicative systems are not computable. We have shown that impredicative systems are usefully modeled as hypersets, and hypersets are Turing-computable. In sum, we have shown that Rosen’s (2000) claim that impredicative systems are not Turing-computable is no longer tenable, which undercuts the justification for his conclusion that models of living systems are not computable. This agrees with the arguments of Chu and Ho (2006), who dispute Rosen’s purported proof that artificial life is impossible, and Wells (2006), who
argued that Rosen’s metabolism-repair systems are computable. Nonetheless, we believe that much of the spirit of Rosen’s work can be maintained. We can see this by realizing that Rosen’s characterization of the distinction between simple and complex systems is multi-dimensional. In distinguishing between simple and complex systems, Rosen argued that simple systems are impoverished when compared to complex systems. Among these ways in which simple systems are impoverished are the following,

(i) complex systems have models that are computable and non-computable, while simple systems only have models that are computable;
(ii) complex systems have models that are impredicative and models that are predicative, while simple systems have only predicative models;
(iii) complex systems can have fractionable and non-fractionable structures, while simple systems can have only fractionable structures;
(iv) complex systems can have both linear chains and closed loops of entailment, while simple systems can only have linear chains of entailment;
(v) complex systems have both syntactic and semantic elements, while simple systems have only syntactic elements. (See Gwinn 2006.)

As we have seen, Rosen’s thinking that (ii) entails (i) is no longer justified. But Rosen’s great insights were in seeing that complex systems have non-fractionable structures, loopy causal entailment, and built-in semantics, and in seeing that this leads to models of complex systems with impredicativities. Future research on Rosen-related themes must focus on the contrasts between simple and complex systems in points (ii) – (v) above.

We contend that the central idea of Rosen’s work is not that artificial life or valuable computational models of autonomous systems are impossible. Very little rides on whether genuine artificial life is possible: we agree with Moreno, Extebarria and Umerez (2008) that the real value of computational research is in what it teaches us about biological autonomy. Instead, the most important point of Rosen’s work is that autonomous systems have a structure that is very different from non-living things and that this structure must be represented in their models. More precisely, autonomous systems are closed to efficient cause and so must have impredicative models. We have shown in this paper that Rosen’s metabolism-repair systems are in fact closed to efficient cause and, so, must have impredicative models. We have also seen that these impredicative models are Turing-computable. We believe that those interested in pursuing research along the lines suggested by Rosen should view this as an opportunity. Computational modeling using hypersets could become an important tool for studying non-fractionable, circularly causal, partly semantic systems such as Rosen’s metabolism-repair system.

References


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