A Hecke Correspondence Theorem for Automorphic Integrals with Symmetric Rational Period Functions on the Hecke Groups

Wendell Ressler

Franklin & Marshall College

January 7, 2009

Background

- Hecke correspondence
- Automorphic Integrals
- Hawkins & Knopp's correspondence

Results

- Rational Period Functions
- Hecke correspondence

Riemann's functional equation

Bernhard Riemann proved that the zeta function satisfies the functional equation

$$\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2} = \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2},$$

using the transformation property of the (elliptic Jacobi) theta function

$$\theta(x) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

- B. Riemann, Über die Ansahl der Primzahlen unter einer gegebenen Grösse, *Monatsber. Berlin. Akad.* (1859), 671-680.

Hecke correspondence

In the 1930s Erich Hecke developed a general theory based on Riemann's ideas.

- The Mellin transform of an entire automorphic form (AF)

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda},$$

  is

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

- The automorphic relation for the AF leads to a functional equation for the Dirichlet series.
- The inverse Mellin transform allows this to be reversed and establishes a correspondence now called **Hecke correspondence**.

Hecke groups

Hecke considered the groups $\langle S_\lambda, T\rangle / \{\pm I\}$ for $\lambda \in \mathbf{R}^+$.

- Generators: $S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$
- Such a group is discrete if and only if $\lambda = \lambda_p = 2\cos(\pi/p)$.
  - E. Hecke, 1936
  - A group supports nontrivial automorphic forms if and only if it it discrete.
- **Hecke groups** are $G_p = \left\langle S_{\lambda_p}, T\right\rangle / \{\pm I\}$ for $p = 3, 4, 5, \ldots$.
  - $G_3 = \Gamma(1)$
  - Group relations: $T^2 = I$, $U^p = I$
    - $U = S_\lambda T$

A generalization

In the 1970s Marvin Knopp introduced a class of functions he called "automorphic integrals" (AIs) on the Hecke groups.

- An AI generalizes an automorphic form by including an additional "period" function in its automorphic relation.
- Knopp's AIs also have Fourier expansions.
- Knopp showed that an AI corresponds to a Dirichlet series with the same functional equation, as long as its period function has poles only at 0.
- What if a period function has other poles?

Automorphic integrals with rational period functions

An **entire automorphic integral** (AI) has a Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda},$$

satisfies some analytic properties and the automorphic relation

$$z^{-2k} F(Tz) = F(z) + q(z).$$

- If $q \equiv 0$, F is an automorphic form.
- $q(z)$ is the **(rational) period function** (RPF) for F.
- The **weight** is $2k$, for $k \in \mathbf{Z}$.
- The **multiplier** is $\gamma = 1$.

Slash operator

Weight $2k$ slash operator:

$$(F \mid M)(z) = (cz+d)^{-2k} F(Mz)$$

- $F(z)$ a complex function
- $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in G_p$
- $F \mid M_1 M_2 = (F \mid M_1) \mid M_2$

Using this notation the automorphic relation is

$$F \mid T = F + q.$$

Rational period function (RPF) relations

The group relation $T^2 = I$ implies that an RPF q satisfies

$$q + q \mid T = 0.$$

The group relation $U^p = I$ implies that q satisfies

$$q + q \mid U + q \mid U^2 + \cdots + q \mid U^{p-1} = 0.$$

These relations characterize RPFs of a given group and weight, independent of the associated automorphic integral(s).

- M. Knopp, 1978

Hecke correspondence, RPFs with nonzero poles

The Mellin transform of an entire automorphic integral with the Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z/\lambda},$$

is

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

- J. Hawkins & M. Knopp, 1992.

The functional equation

$\Phi(s)$ is meromorphic and satisfies the functional equation

$$\Phi(2k-s)+\Phi(s)=R(s),$$

where $R(s)$ is a meromorphic function Hawkins and Knopp called the **remainder term**.

- $R(s)$ depends only on the nonzero poles of the RPF.
- $R(s)$ satisfies the (first) relation

  $$R(2k-s)-R(s)=0,$$

  which is equivalent to the fact that the RPF satisfies the first relation

  $$q+q \mid T=0.$$

The second relation?

Hawkins and Knopp proved their correspondence on subgroups of the Hecke groups which have only one group relation.

- The RPFs and remainder terms satisfy only one relation.
- What about RPFs on the full Hecke groups - with a second relation?

Rational period functions

The details of the correspondence depend on the structure of the RPFs.

- The characterization of rational period functions on the Hecke groups falls into three cases.
  1. Hecke-symmetric RPFs, k odd
  2. Non Hecke-symmetric RPFs
  3. Hecke-symmetric RPFs, k even
- Case 1 is known.
- Cases 2 and 3 are open.

Hecke symmetry

- Poles of RPFs are hyperbolic fixed points of the Hecke group.
- If α is hyperbolic, we define the **Hecke conjugate** of α to be the other point fixed by elements of stab(α).
- An RPF has **Hecke symmetry** if each of its poles is the Hecke conjugate of another pole of the same RPF.

Characterize Hecke-symmetric RPFs, k odd

Any RPF in this case has the form

$$q(z) = \sum_{\ell=1}^{L} d_\ell D_\ell^{-k/2} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} \frac{(\alpha - \alpha')^k}{(z-\alpha)^k (z-\alpha')^k} + c_0 q_0(z).$$

- The $\mathcal{A}_\ell$ are equivalence classes of $\mathbf{Z}[\lambda]$-binary quadratic forms.
- D_ℓ is the discriminant of the quadratic forms in $\mathcal{A}_\ell$.
- $\mathcal{Z}_{\mathcal{A}_\ell}$ is the set of "simple" numbers associated with $\mathcal{A}_\ell$.
- α' is the Hecke conjugate of α.
- q_0 is an RPF with a pole only at zero.
- Ressler, 2001

The second relation, rational period functions

We exhibit the fact that q satisfies the second relation by writing

$$q(z) = \sum_{\ell=1}^{L} c_\ell \sum_{j=2}^{p} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \bigcap I_j} \left(Q_\alpha(z)^{-k} - Q_\alpha^{-k}(z) \mid U^{p-j+1} \right) + c_0 q_0(z).$$

- The $I_j = [U^{p-j+2}(0), U^{p-j+1}(0)\,)$ are disjoint intervals that partition the real axis.
- $Q_\alpha(z)^{-k} = \frac{D^{-k/2}(\alpha-\alpha')^k}{(z-\alpha)^k(z-\alpha')^k}$
- Any function of the form $r - r \mid U^m = r \mid (I - U^m)$ for $m \in \mathbf{Z}^+$ satisfies the second relation

$$q + q \mid U + q \mid U^2 + \cdots + q \mid U^{p-1} = 0.$$

The remainder term

The corresponding remainder term is

$$R(s) = -\sum_{\ell=1}^{L} c_\ell D_\ell^{-k/2} \times \sum_{j=2}^{p} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \bigcap I_j} \left(R\left(s; \alpha, \alpha'\right) - R\left(s; U^{j-1}\alpha, U^{j-1}\alpha'\right)\right),$$

where

$$R(s; a, b) = (a-b)^k \int_0^\infty \frac{y^s}{(iy-a)^k (iy-b)^k} \frac{dy}{y}$$
$$= \alpha^{s-k} B(2k-s, s-k)_2F_1\left[k, 1-k; k-s+1; \frac{\alpha'}{\alpha'-\alpha}\right]$$
$$+ \left(\alpha'\right)^{s-k} B(s, k-s)_2F_1\left[k, 1-k; s-k+1; \frac{\alpha'}{\alpha'-\alpha}\right].$$

The second relation, remainder term

Define the mapping

$$\rho\left(R\left(s;a,b\right)\right)=-R\left(2k-s;a-\lambda,b-\lambda\right).$$

Then $R(s)$ satisfies the (second) relation

$$R+\rho(R)+\rho^2(R)+\cdots+\rho^{p-1}(R)=0,$$

which is equivalent to the fact that the RPF satisfies the second relation

$$q+q\mid U+q\mid U^2+\cdots+q\mid U^{p-1}=0.$$

Thank You

Preprints and slides for this talk are available at `http://edisk.fandm.edu/wendell.ressler/`.

- W. Ressler, On Binary Quadratic Forms and the Hecke groups, to appear, *International Journal of Number Theory*.
- W. Ressler, A Hecke Correspondence Theorem for Automorphic Integrals with Symmetric Rational Period Functions on the Hecke Groups, preprint.