Mellin Transforms of Rational Period Functions

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Transform the theta series
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Riemann's paper

B. Riemann, Über die Ansahl der Primzahlen unter einer gegebenen Grösse, *Monatsber. Berlin. Akad.* (1859), 671-680.

- Outlined a plan for proving the Prime Number Theorem.
- Proved the functional equation for the zeta function.
- Made a conjecture about the location of the nontrivial zeros of the zeta function - the Riemann hypothesis.

Georg Friedrich Bernhard Riemann (1826-1866)

Riemann Zeta function

Definition: Let $s = \sigma + it \in \mathbf{C}$. For $\sigma > 1$ we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- First studied by Euler.
- Euler: Product formula holds for $s > 1$.
- Riemann: First considered s to be a complex number.
- The product converges absolutely for $\sigma > 1$ $\implies \zeta(s)$ has no zeros for $\sigma > 1$.

Analytic form of the Fundamental Theorem of Arithmetic

For $s > 1$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$
$$= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$
$$= \sum_{\substack{p_j \text{ prime} \\ k_j \geq 0}} \frac{1}{\left(p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots\right)^s}.$$

Every positive integer is represented uniquely in the last sum.

Functional equation for $\zeta(s)$

For $s \in \mathbf{C}$ the Riemann zeta function satisfies

$$\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2} = \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}.$$

- Euler conjectured this, verified it for integral values of s.
- $\Gamma(z)$ is the Gamma function, given by $\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$ for $\Re(z) > 0$.
- Provides an analytic continuation of $\zeta(s)$ to $\sigma < 0$.
- Riemann's proof uses the transformation property of the (elliptic Jacobi) theta function.

Theta function

Definition: For $x > 0$ we have

$$\begin{aligned} \theta(x) &= \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \\ &= 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 x} \\ &= 1 + 2\Psi(x). \end{aligned}$$

Theta transformation formula (Jacobi)

$$\theta\left(\frac{1}{x}\right) = x^{1/2}\theta(x)$$

$$\implies \Psi\left(\frac{1}{x}\right) = x^{1/2}\Psi(x) + \frac{1}{2}\left(\sqrt{x} - 1\right)$$

Riemann's proof of the functional equation, I

Start with the integral formula for $\Gamma\left(\frac{s}{2}\right)$,

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t} t^{s/2} \frac{dt}{t}.$$

Let $t = \pi n^2 x$ for $n \in \mathbf{Z}^+$, and rearrange to get

$$n^{-s}\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = \int_0^\infty e^{-\pi n^2 x} x^{s/2} \frac{dx}{x}.$$

Sum as n runs from 1 to ∞, so

$$\begin{aligned}\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} &= \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 x}\right) x^{s/2} \frac{dx}{x} \\ &= \int_0^\infty \Psi(x) x^{s/2} \frac{dx}{x}.\end{aligned}$$

Riemann's proof of the functional equation, II

Invert part of the integral and use the theta transformation formula to calculate

$$\begin{aligned}\int_0^1 \Psi(x)x^{s/2}\frac{dx}{x} &= \int_1^\infty \Psi\left(\frac{1}{x}\right)x^{-s/2}\frac{dx}{x}\\ &= \frac{1}{2}\int_1^\infty \left(\sqrt{x}-1\right)x^{-s/2}\frac{dx}{x} + \int_1^\infty \Psi(x)x^{\frac{1-s}{2}}\frac{dx}{x}\\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \Psi(x)x^{\frac{1-s}{2}}\frac{dx}{x}.\end{aligned}$$

Riemann's proof of the functional equation, III

Combine the two integral expressions to get

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \Psi(x)\left\{x^{\frac{1-s}{2}} + x^{\frac{s}{2}}\right\}\frac{dx}{x}.$$

The right hand side is unchanged if we replace s by $1-s$, so

$$\zeta(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2} = \zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}.$$

□

A more general context for Riemann's work, I

- The theta function $\theta(-iz) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z}$ is a **modular form** of weight $\frac{1}{2}$.
  - Invariant under translation by 2: $z \rightarrow z + 2$
  - Almost invariant under inversion: $z \rightarrow \frac{-1}{z}$
- $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is a **Dirichlet series** (a.k.a., *L*-series).
- $\Gamma(s)$ is the **Mellin transform** of e^{-t}.

A more general context for Riemann's work, II

The Mellin transform of a Modular form is a Dirichlet series.

- The transformation property of the Modular form determines a functional equation for the Dirichlet series.
- This is reversible, using the inverse Mellin transform.

The modular group

The linear fractional transformations S and T generate the **modular group** $\Gamma(1)$.

- $S : z \to z + 1$ and $T : z \to \frac{-1}{z}$
- Group relations: $T^2 = I$, $(ST)^3 = I$
- Modular forms are almost invariant with respect to $\Gamma(1)$.
- We view $\Gamma(1)$ as the group of matrices $PSL(2, \mathbf{Z})$ acting on $\mathbf{C}$ as linear fractional transformations.
  - $PSL(2, \mathbf{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \mid ad - bc = 1 \right\} / \{\pm I\}$
  - $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) z = \frac{az+b}{cz+d}$

Modular forms

A **modular form** satisfies

$$F\left(Sz\right) = F(z),$$

and

$$F\left(Tz\right) = \gamma z^{k} F(z).$$

- γ is a constant of modulus 1, the **multiplier**.
- k is a real number, the **weight**.

Dirichlet series

A **Dirichlet series** has the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

- The a_n are complex constants.
- $s = \sigma + it$ is a complex variable.
- The series converges in some right half-plane $\sigma > \sigma_0$.
- Johann Peter Gustav Lejeune Dirichlet (1805-1859)
  - Primes in arithmetic progression, 1837

The Mellin transform and its inverse

The **Mellin transform** of $f(t)$ is

$$\phi(s) = \int_0^\infty f(t) t^s \frac{dt}{t}.$$

- A modification of the bilateral Laplace transform.
- Robert Hjalmar Mellin (1854-1933)

The **inverse Mellin transform** of $\phi(s)$ is

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) t^{-s} ds.$$

- The line of integration is a vertical line with real part c.

Hecke correspondence

In the 1930s Erich Hecke developed a general theory based on Riemann's ideas.

- The Mellin transform of any automorphic form (AF) is a Dirichlet series.
- The automorphic relation for the AF leads to the functional equation for the Dirichlet series.
- The inverse Mellin transform allows this to be reversed and establishes a correspondence now called **Hecke correspondence**.

Hecke groups

Consider the class of groups $\langle S_\lambda, T\rangle / \{\pm I\}$ for $\lambda \in \mathbf{R}^+$.

- Generators: $S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$
- The group is discrete if and only if $\lambda = \lambda_p = 2\cos(\pi/p)$.
  - E. Hecke, 1936
  - A **discrete** group contains no convergent sequence of distinct elements.
  - A group supports nontrivial automorphic forms if and only if it it discrete.
- **Hecke groups** are $G_p = \langle S_\lambda, T\rangle / \{\pm I\}$ for $p = 3, 4, 5, \ldots$.
  - $G_3 = \Gamma(1)$
  - Group relations: $T^2 = I$, $U^p = I$
    - $U = S_\lambda T$

Automorphic forms

An **automorphic form** satisfies

$$F\left(S_\lambda z\right) = F(z),$$

and

$$F\left(Tz\right) = \gamma z^k F(z). \tag{1}$$

- γ is a constant of modulus 1, called the **multiplier**.
- k is a real number, called the **weight**.
- (1) is the **automorphic relation** for F.
- $T^2 = I \implies \gamma^2(-1)^k = 1.$

Fourier expansion

An automorphic form F has the Fourier expansion

$$F(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z/\lambda}.$$

- $z \in \mathcal{H}$ (the upper half-plane)
- If $n_0 \geq 0$, F is an **entire** automorphic form.
- If $n_0 > 0$, F is a **cusp** automorphic form.

Hecke's direct theorem, I

The Mellin transform of $F(z)$ is

$$\Phi(s) = \int_0^\infty F(iy) y^s \frac{dy}{y}.$$

We use the Fourier expansion of $F(z)$ to calculate

$$\begin{aligned}\Phi(s) &= \int_0^\infty \left(\sum_{n=1}^\infty a_n e^{2\pi i n z/\lambda} \right) y^s \frac{dy}{y} \\ &= \sum_{n=1}^\infty a_n \int_0^\infty e^{2\pi i n z/\lambda} y^s \frac{dy}{y} \\ &= \left(\frac{2\pi}{\lambda} \right)^{-s} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^s}\end{aligned}$$

- $\phi(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}$ is the **Dirichlet series associated with** F.

Hecke's direct theorem, II

Following Riemann, we invert part of the integral and use the automorphic relation to calculate

$$\int_0^1 F(iy)y^s \frac{dy}{y} = \int_1^\infty F\left(\frac{-1}{iy}\right) y^{-s} \frac{dy}{y}$$
$$= \gamma i^k \int_1^\infty F(iy)\, y^{k-s} \frac{dy}{y}.$$

Hecke's direct theorem, III

We combine the two integral expressions to get

$$\Phi(s) = \int_1^\infty F(iy) \left\{ y^s + \gamma i^k y^{k-s} \right\} \frac{dy}{y}.$$

Since $\gamma^2 i^{2k} = 1$, we have that Φ satisfies the functional equation

$$\Phi(k-s) = \gamma i^k \Phi(s),$$

or

$$\left(\frac{2\pi}{\lambda}\right)^{-(k-s)} \Gamma(k-s)\phi(k-s) = \gamma i^k \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\phi(s).$$

Hecke's converse theorem

Hecke proved a converse to his "direct" theorem.

- Hecke's direct theorem also showed that the Mellin transform of an automorphic form has certain analytic properties.
- Hecke used these properties along with the inverse Mellin transform to prove the converse.
- Hecke showed that a Dirichlet series with a functional equation leads to an automorphic form.

Another generalization

In the 1970s Marvin Knopp introduced a class of functions he called automorphic integrals.

- An automorphic integral generalizes an automorphic form by including an additional "period" function in its automorphic relation.
- Knopp showed that an automorphic integral satisfies a similar "Hecke correspondence" with Dirichlet series, as long as its period function has poles only at 0.
- What if a period function has other poles?

Automorphic integrals with rational period functions

An **automorphic integral** satisfies

$$F\left(S_\lambda z\right) = F(z),$$

and

$$z^{-2k} F\left(Tz\right) = F(z) + q(z).$$

- If $q \equiv 0$, F is an automorphic form.
- $q(z)$ is the **(rational) period function** for F.
- The weight is $2k$, for $k \in \mathbf{Z}$.
- The multiplier is $\gamma = 1$.

Slash operator

Weight $2k$ slash operator:

$$(F \mid M)(z) = (cz + d)^{-2k} F(Mz)$$

- $F(z)$ a complex function
- $M = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in G_p$
- $F \mid M_1 M_2 = (F \mid M_1) \mid M_2$

Using this notation the automorphic relation is

$$F \mid T = F + q.$$

Rational period function relations

The group relation $T^2 = I$ implies that a rational period function (RPF) q satisfies

$$q + q \mid T = 0.$$

The group relation $U^p = I$ implies that q satisfies

$$q + q \mid U + q \mid U^2 + \cdots + q \mid U^{p-1} = 0.$$

These relations characterize rational period functions of a given group and weight, independent of the associated automorphic integral(s).

- Marvin Knopp, 1978

Fourier expansion

An automorphic integral F has the Fourier expansion

$$F(z) = \sum_{n=n_0}^{\infty} a_n e^{2\pi i n z/\lambda}.$$

- $z \in \mathcal{H}$ (the upper half-plane)
- If $n_0 \geq 0$, F is an **entire** automorphic integral.
- If $n_0 > 0$, F is a **cusp** automorphic integral.
- This is the same expression as the Fourier expansion of an automorphic form.

The Mellin Transform of an automorphic integral, I

The Mellin transform of $F(z)$ is

$$\Phi(s) = \int_0^\infty F(iy) y^s \frac{dy}{y} = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\phi(s).$$

- $\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is the Dirichlet series associated with F.
- This is the same expression as the Mellin transform of an automorphic form.

The Mellin Transform of an automorphic integral, II

Following Riemann and Hecke, we invert part of the integral and use the automorphic relation to calculate

$$\int_0^1 F(iy)y^s\frac{dy}{y} = \int_1^\infty F\left(\frac{-1}{iy}\right) y^{-s}\frac{dy}{y}$$
$$= i^{2k}\int_1^\infty F\left(iy\right) y^{2k-s}\frac{dy}{y} + i^{2k}\int_1^\infty q\left(iy\right) y^{2k-s}\frac{dy}{y}.$$

The Mellin Transform of an automorphic integral, III

We combine the two integral expressions to get

$$\Phi(s) = \int_1^\infty F(iy)\left\{y^s + i^{2k}y^{2k-s}\right\}\frac{dy}{y} + i^{2k}\int_1^\infty q\left(iy\right)y^{2k-s}\frac{dy}{y}$$
$$= D(s) + E(s).$$

Since $i^{2k} = i^{-2k}$, $D(s)$ satisfies the functional equation

$$D(2k - s) = i^{2k}D(s).$$

The Mellin Transform of an automorphic integral, IV

Since $D(2k-s) - i^{2k}D(s) = 0$ we have

$$\begin{aligned}\Phi(2k-s) - i^{2k}\Phi(s) &= E(2k-s) - i^{2k}E(s) \\ &= R(s).\end{aligned}$$

- $R(s)$ is the **remainder term** for the Dirichlet series functional equation.
- This is meaningless, unless we calculate an explicit expression and/or properties for $R(s)$.

Convergence problem

The remainder term is

$$R(s) = E(2k-s) - i^{2k}E(s)$$
$$= i^{2k}\int_1^\infty q\,(iy)\,y^s\frac{dy}{y} - \int_1^\infty q\,(iy)\,y^{2k-s}\frac{dy}{y}.$$

- Problem: The integrals have no common region of convergence.
- The integrals are partial Mellin transforms of rational functions.

Split $E(s)$

We can write the rational period function q as

$$q(z) = c_0 q_0(z) + q^*(z).$$

- $q_0(z)$ has poles only at 0 of order at most $2k$.
- $q^*(z)$ has nonzero poles of order k.
- q_0 and q^* are both RPFs by themselves.

Then

$$E(s) = E^0(s) + E^*(s)$$

- $E^0(s) = i^{2k} \int_1^\infty q_0\left(iy\right) y^{2k-s} \frac{dy}{y}$
- $E^*(s) = i^{2k} \int_1^\infty q^*\left(iy\right) y^{2k-s} \frac{dy}{y}$

A functional equation for $E^0(s)$

A calculation shows that

$$E^0(s) = \begin{cases} a_0 \left(\frac{i^{2k}}{s-2k} - \frac{1}{s} \right), & \text{if } 2k \neq 2, \\ a_0 \left(\frac{i^{2k}}{s-2} - \frac{1}{s} \right) + \frac{b_1 i}{s-1}, & \text{if } 2k = 2. \end{cases}$$

Thus a functional equation for E^0 is

$$E^0(2k-s) - i^{2k} E^0(s) = 0.$$

- This also provides a meromorphic continuation of E^0 to $\mathbf{C}$.

Convergence OK

Since $E^0(2k - s) - i^{2k} E^0(s) = 0$ we have

$$\begin{aligned} R(s) &= E^*(2k - s) - i^{2k} E^*(s) \\ &= i^{2k} \int_1^\infty q^*(iy)\, y^s \frac{dy}{y} - \int_1^\infty q^*(iy)\, y^{2k-s} \frac{dy}{y}. \end{aligned}$$

- No convergence problem: Both integrals converge if $0 < \sigma < 2k$.

The first relation, remainder term

$R(s) = E^*(2k - s) - i^{2k} E^*(s)$ satisfies the (first) relation

$$R(2k - s) + i^{2k} R(s) = 0.$$

- Equivalent to the first relation for q^*.
- First appeared in a paper by J. Hawkins and M. Knopp, 1992.

The remainder term

We invert the second integral and use the fact that q^* satisfies the first relation to calculate that

$$-\int_1^\infty q^*(iy)\, y^{2k-s} \frac{dy}{y} = i^{2k} \int_0^1 q^*(iy)\, y^s \frac{dy}{y}.$$

Then

$$\begin{aligned} R(s) &= i^{2k} \int_1^\infty q^*(iy)\, y^s \frac{dy}{y} - \int_1^\infty q^*(iy)\, y^{2k-s} \frac{dy}{y} \\ &= i^{2k} \int_0^\infty q^*(iy)\, y^s \frac{dy}{y}. \end{aligned}$$

- The remainder term is a Mellin transform of q^*.
- We need information about q^*.

Cases

- The characterization of rational period functions on the Hecke groups falls into three cases.
  1. Hecke-symmetric RPFs, k odd
  2. Non Hecke-symmetric RPFs
  3. Hecke-symmetric RPFs, k even
- Case 1 is known.
- Cases 2 and 3 are open.

Characterize Hecke-symmetric RPFs, k odd

Any RPF in this case has the form

$$q(z) = \sum_{\ell=1}^{L} d_\ell D_\ell^{-k/2} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} \frac{(\alpha - \alpha')^k}{(z-\alpha)^k (z-\alpha')^k} + c_0 q_0(z).$$

- The $\mathcal{A}_\ell$ are equivalence classes of $\mathbf{Z}[\lambda]$-binary quadratic forms.
- $\mathcal{Z}_{\mathcal{A}_\ell}$ are "simple" numbers associated with $\mathcal{A}_\ell$.
- α' is the "Hecke conjugate" of α.
- D_ℓ is the discriminant of the quadratic forms in $\mathcal{A}_\ell$.
- Ressler, 2001

The remainder term for Hecke-symmetric RPFs, k odd

In this case we have

$$R(s) = -\sum_{\ell=1}^{L} d_\ell D_\ell^{-k/2} i^s$$

$$\times \sum_{\alpha \in Z_{\mathcal{A}_\ell}} \{\alpha^{s-k} B(2k-s, s-k)_2F_1\left[k, 1-k; k-s+1; \frac{\alpha'}{\alpha'-\alpha}\right]$$

$$+ (\alpha')^{s-k} B(s, k-s)_2F_1\left[k, 1-k; s-k+1; \frac{\alpha'}{\alpha'-\alpha}\right]\}.$$

- $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the beta function.
- ${}_2F_1[a, b; c; s]$ is Gauss' hypergeometric function.
- We can use this to verify that $R(2k-s) + i^{2k} R(s) = 0$.

The second relation, rational period functions

A calculation shows that any function of the form

$$r - r \mid U^m = r \mid (I - U^m)$$

for $m \in \mathbf{Z}^+$ satisfies the second relation

$$q + q \mid U + q \mid U^2 + \cdots + q \mid U^{p-1} = 0.$$

We exhibit the fact that q satisfies the second relation by writing

$$q^*(z) = \sum_{\ell=1}^{L} c_\ell \sum_{j=2}^{p} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \bigcap I_j} \left(Q_\alpha(z)^{-k} - Q_\alpha^k(z) \mid U^{p-j+1} \right).$$

- $Q_\alpha(z)^{-k} = \frac{D^{-k/2}(\alpha-\alpha')^k}{(z-\alpha)^k(z-\alpha')^k}$

The remainder term

The corresponding remainder term is

$$R(s) = -\sum_{\ell=1}^{L} c_\ell D_\ell^{-k/2} \times \sum_{j=2}^{p} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \bigcap I_j} \left(R\left(s;\alpha,\alpha'\right) - R\left(s; U^{j-1}\alpha, U^{j-1}\alpha'\right)\right),$$

where

$$R(s;a,b) = (a-b)^k \int_0^\infty \frac{y^s}{(iy-a)^k (iy-b)^k} \frac{dy}{y}.$$

The second relation, remainder term

Define the mapping

$$\rho\left(R\left(s;a,b\right)\right)=-R\left(2k-s;a-\lambda,b-\lambda\right).$$

Then $R(s)$ satisfies the (second) relation

$$R+\rho(R)+\rho^2(R)+\cdots+\rho^{p-1}(R)=0.$$

Preprints

Preprints are available on my web site.

- W. Ressler, On Binary Quadratic Forms and the Hecke groups, to appear, *International Journal of Number Theory*.
- W. Ressler, A Hecke Correspondence Theorem for Automorphic Integrals with Symmetric Rational Period Functions on the Hecke Groups, preprint.