A NOTE ON SIEGEL'S PROOF OF HAMBURGER'S THEOREM

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Dedicated to Leon Ehrenpreis, whose questions have engaged many minds, including that of Carl Ludwig Siegel

In 1921 Hamburger [3] proved the following result concerning the Riemann zeta function $\zeta(s)$:

Theorem 1 (Hamburger). *Let $G(s)$ be an entire function of finite order of $s = \sigma + it$, $P(s)$ a polynomial, and $f(s) = \frac{G(s)}{P(s)}$, with*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converging absolutely for $\sigma > 1$. Let

$$f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = g(1-s)\Gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}, \tag{1}$$

where

$$g(1-s) = \sum_{n=1}^{\infty} \frac{b_n}{n^{1-s}}$$

is absolutely convergent for $\sigma < -\alpha < 0$ ($\alpha > 0$). Then $f(s) = a_1\zeta(s)$.

Later that same year Siegel wrote a letter to Hamburger [5] in which he gave a (by his own admission) simpler and shorter proof of the result. Siegel also showed that Hamburger's assumption on the absolute convergence of $f(s)$ for

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$\sigma > 1$ can be weakened to $\sigma > 2-\theta$, $\theta > 0$. In this note we relax this condition completely by assuming only that $f(s)$ converges somewhere. What's more, our proof essentially follows that of Siegel. The main difference is that we use a more general integral transform which was recently introduced by Heath, Flood and the current authors [2]. This transform, $I(t) = I_\rho[F;t] = \int_0^\infty F(x)x^\rho e^{-\pi t^2 x}dx$, with ρ a real parameter, reduces to the one used by Siegel when $\rho = 0$. Our theorem also involves a more general functional equation and extends a result found in [1] and [2]. We shall prove the following

Theorem 2. *Let $G(s)$ be an entire function of finite order of $s = \sigma + it$, $P(s)$ a polynomial, and $f(s) = \frac{G(s)}{P(s)}$, with*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converging somewhere. Let

$$f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = g(2k-s)\Gamma\left(k-\frac{s}{2}\right)\pi^{-\left(k-\frac{s}{2}\right)}, \tag{2}$$

where k is an arbitrary real number and

$$g(2k-s) = \sum_{n=1}^{\infty} \frac{b_n}{n^{2k-s}}$$

is convergent somewhere. Then $f(s) = a_1\zeta(s)$ for $k = \frac{1}{2}$, but $f(s) \equiv 0$ for $k \neq \frac{1}{2}$.

Proof. A Dirichlet series which converges somewhere must also converge absolutely somewhere. So let us assume that $f(s)$ converges absolutely for $\sigma > \alpha$ and that $g(2k-s)$ converges absolutely for $\sigma < \beta$. Without loss of generality we also suppose that $\alpha \geq 0$ and $\beta \leq 2k$. We first apply the inverse Mellin transform to the functional equation (2). For $x > 0$ we have

$$\begin{aligned} S(x) &= \frac{1}{2\pi i}\int_{\alpha+1-i\infty}^{\alpha+1+i\infty} f(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}x^{-\frac{s}{2}}ds \\ &= 2\sum_{n=1}^{\infty} a_n \frac{1}{2\pi i}\int_{\frac{\alpha+1}{2}-i\infty}^{\frac{\alpha+1}{2}+i\infty} \Gamma(s)(\pi n^2 x)^{-s}ds \\ &= 2\sum_{n=1}^{\infty} a_n e^{-\pi n^2 x} \end{aligned}$$

and also

$$\begin{aligned} S(x) &= \frac{1}{2\pi i}\int_{\alpha+1-i\infty}^{\alpha+1+i\infty} g(2k-s)\Gamma\left(k-\frac{s}{2}\right)\pi^{-\left(k-\frac{s}{2}\right)}x^{-\frac{s}{2}}ds \\ &= \frac{1}{2\pi i}\int_{\beta-1-i\infty}^{\beta-1+i\infty} g(2k-s)\Gamma\left(k-\frac{s}{2}\right)\pi^{-\left(k-\frac{s}{2}\right)}x^{-\frac{s}{2}}ds + \sum_{j=1}^{N} R_j, \end{aligned}$$

where we have used Stirling's formula and a theorem of Phragmén-Lindelöf to move the line of integration to $\sigma = \beta - 1$. Here each R_j, $1 \le j \le N$, is the residue of the pole at s_j of the integrand in the region $\beta - 1 < \sigma < \alpha + 1$. In fact

$$\sum_{j=1}^{N} R_j = \sum_{j=1}^{N} x^{-\frac{s_j}{2}} Q_j(\log x) = Q(x),$$

where each Q_j is a polynomial. Replacing s with $2k - 2s$, inserting the Dirichlet series for $g(2s)$ and interchanging the sum and integral we get

$$\begin{aligned} S(x) &= \frac{2}{x^k} \sum_{n=1}^{\infty} b_n \left\{ \frac{1}{2\pi i} \int_{\frac{-\beta+2k+1}{2}-i\infty}^{\frac{-\beta+2k+1}{2}+i\infty} \Gamma(s) \left(\frac{\pi n^2}{x} \right)^{-s} ds \right\} + Q(x) \\ &= \frac{2}{x^k} \sum_{n=1}^{\infty} b_n e^{\frac{-\pi n^2}{x}} + Q(x). \end{aligned}$$

Hence the functional equation (2) implies the modular relation

$$2 \sum_{n=1}^{\infty} a_n e^{-\pi n^2 x} = \frac{2}{x^k} \sum_{n=1}^{\infty} b_n e^{\frac{-\pi n^2}{x}} + Q(x), \tag{3}$$

for $x > 0$.

Next we multiply (3) by $x^{\rho} e^{-\pi t^2 x}$ (t and ρ fixed with $t > 0$ and $\rho > \frac{\alpha}{2} - 1$) and integrate with respect to x from 0 to ∞. The left side of (3) becomes

$$\begin{aligned} \int_0^{\infty} \left(2 \sum_{n=1}^{\infty} a_n e^{-\pi n^2 x} \right) x^{\rho} e^{-\pi t^2 x} dx &= 2 \sum_{n=1}^{\infty} a_n \int_0^{\infty} e^{-\pi (n^2 + t^2) x} x^{\rho} dx \\ &= \frac{2\Gamma(\rho+1)}{\pi^{\rho+1}} \sum_{n=1}^{\infty} \frac{a_n}{(t^2 + n^2)^{\rho+1}}. \end{aligned}$$

The above steps are valid since $\rho > \frac{\alpha}{2} - 1 \ge -1$. The right side of (3) gives

$$\begin{aligned} \int_0^{\infty} \left(\frac{2}{x^k} \sum_{n=1}^{\infty} b_n e^{\frac{-\pi n^2}{x}} \right) x^{\rho} e^{-\pi t^2 x} dx + \int_0^{\infty} (Q(x))\, x^{\rho} e^{-\pi t^2 x} dx \\ = 2 \sum_{n=1}^{\infty} b_n \int_0^{\infty} e^{-\pi t^2 x - \frac{\pi n^2}{x}} \frac{dx}{x^{k-\rho}} + H(t). \end{aligned}$$

The interchange of the sum and integral is valid for all values of ρ and k. Note that

$$\begin{aligned} H(t) &= \sum_{j=1}^{N} t^{s_j - 2 - 2\rho} \int_0^{\infty} x^{\rho - \frac{s_j}{2}} Q_j(\log x - 2\log t) e^{-\pi x} dx \\ &= \sum_{j=1}^{N} t^{s_j - 2 - 2\rho} H_j(\log t), \end{aligned}$$

with H_j a polynomial. Thus, it is apparent that $H(t)$ is analytic in the slit plane $\mathbf{C} \setminus (-\infty, 0]$. Here we have used $\rho > \frac{\alpha}{2} - 1$ and $\mathrm{Re}(s_j) \leq \alpha$ (which follows from the absolute convergence of $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ for $\sigma > \alpha$) to insure the convergence of the integral. We have

$$\frac{2\Gamma(\rho+1)}{\pi^{\rho+1}} \sum_{n=1}^{\infty} \frac{a_n}{(t^2+n^2)^{\rho+1}} = 2 \sum_{n=1}^{\infty} b_n \int_0^{\infty} e^{-\pi t^2 x - \frac{\pi n^2}{x}} \frac{dx}{x^{k-\rho}} + H(t) \tag{4}$$

for any real number k, with $t > 0$ and $\rho > \frac{\alpha}{2} - 1$.

We now let $\rho = k + \ell - \frac{1}{2}$, $\ell = 0, 1, 2, \ldots$, in (4) and use the formula

$$\int_0^{\infty} e^{-a^2 x - \frac{b^2}{x}} \frac{x^{\ell}}{\sqrt{x}} dx = \frac{\sqrt{\pi}}{a^{2\ell+1}} e^{-2ab} \sum_{p=0}^{\ell} \frac{(2\ell-p)!}{(\ell-p)!p!} 4^{p-\ell} (ab)^p, \quad a > 0,\, b \geq 0,$$

found by differentiating (and then dividing by $-2a$) ℓ times with respect to a in the corresponding formula for $\ell = 0$, to obtain

$$\sum_{n=1}^{\infty} \frac{a_n t^{2\ell+1}}{(t^2+n^2)^{k+\ell+\frac{1}{2}}} - \frac{C}{2} t^{2\ell+1} H(t) = C \sum_{n=1}^{\infty} b_n \sum_{p=0}^{\ell} A_{\ell,p} (nt)^p e^{-2\pi n t}, \tag{5}$$

valid for $k > -\ell + \frac{\alpha-1}{2}$. Here $C = C_{k,\ell} = \frac{\pi^{k+\ell+\frac{1}{2}}}{\Gamma(k+\ell+\frac{1}{2})}$ and $A_{\ell,p} = \frac{(2\ell-p)!}{(\ell-p)!p!}(4\pi)^{p-\ell}$. Both sums in (5) converge uniformly on compact subsets of the right half t-plane (for any k), so they are both analytic in the right half-plane. Next, we multiply both sides of (5) by $(t-mi)^{k+\ell+\frac{1}{2}}, m \in \mathbf{Z}^+$, and take the limit as t approaches mi from the right half-plane along the path $t = mi + u$, $u > 0$. After multiplication by $(t-mi)^{k+\ell+\frac{1}{2}}$, the sum on the left side of (5) converges uniformly on $\{t = mi + u \mid 0 \leq u \leq \delta, \delta > 0\}$, so we may take the limit inside the summation. This gives us that

$$a_m \frac{(mi)^{2\ell+1}}{(2mi)^{k+\ell+\frac{1}{2}}} = C \lim_{u \to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n \sum_{p=0}^{\ell} A_{\ell,p} n^p (mi+u)^p e^{-2\pi n u}, \tag{6}$$

where we used the analyticity of $H(t)$ at mi. Now,

$$\begin{aligned} &\lim_{u \to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n \sum_{p=0}^{\ell} A_{\ell,p} n^p (mi+u)^p e^{-2\pi n u} \\ &\qquad = \lim_{u \to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n \sum_{j=0}^{\ell} u^j \sum_{p=j}^{\ell} A_{\ell,p} n^p \binom{p}{j} (mi)^{p-j} e^{-2\pi n u} \\ &\qquad = \lim_{u \to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n \sum_{p=0}^{\ell} A_{\ell,p} n^p (mi)^p e^{-2\pi n u}, \end{aligned}$$

since the existence of the limit implies that only the terms with the smallest power of u survive. Then, by an argument involving l'Hôpital's rule, the existence of the limit also implies that only the term with the largest power of n can survive. Hence,

$$\lim_{u\to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n \sum_{p=0}^{\ell} A_{\ell,p} n^p (mi)^p e^{-2\pi n u} = \lim_{u\to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n A_{\ell,\ell} n^\ell (mi)^\ell e^{-2\pi n u},$$

and therefore

$$a_m \frac{(mi)^{\ell+1}}{(2mi)^{k+\ell+\frac{1}{2}}} = C \lim_{u\to 0^+} u^{k+\ell+\frac{1}{2}} \sum_{n=1}^{\infty} b_n n^\ell e^{-2\pi n u}. \tag{7}$$

(Note that $A_{\ell,\ell} = 1$.) But the limit on the right side is independent of m and so we get that $a_m = a_1 m^{k-\frac{1}{2}}$ for $m \in \mathbf{Z}^+$. That is, $f(s) = a_1\zeta(s-k+\frac{1}{2})$, where k is *any* real number.

We now claim that for f to be nontrivial we must have that $k = \frac{1}{2}$. We multiply both sides of (5) by $(t+mi)^{k+\ell+\frac{1}{2}}, m \in \mathbf{Z}^+$, and take the limit as t approaches $-mi$ along $t = -mi + u$, $u > 0$. This calculation, coupled with (7), implies that $\frac{a_m}{(-i)^{k-\frac{1}{2}}} = \frac{a_m}{i^{k-\frac{1}{2}}}$. So either $f \equiv 0$ or $k - \frac{1}{2}$ is an even integer. Hence, for f to be nontrivial we must have that $k = \frac{1}{2} + 2\mu$, $\mu \in \mathbf{Z}$, and $f(s) = a_1\zeta(s-2\mu)$. From our hypothesis we know that

$$a_1\zeta(s-2\mu)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = g(4\mu+1-s)\Gamma\left(2\mu+\frac{1}{2}-\frac{s}{2}\right)\pi^{-\left(2\mu+\frac{1}{2}-\frac{s}{2}\right)}.$$

And from the functional equations for $\zeta(s)$ and $\Gamma(s)$ used in conjunction we have that

$$\zeta(s-2\mu)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = r_\mu(s)\zeta(2\mu+1-s)\Gamma\left(2\mu+\frac{1}{2}-\frac{s}{2}\right)\pi^{-\left(2\mu+\frac{1}{2}-\frac{s}{2}\right)},$$

where $r_\mu(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\mu+\frac{1}{2}-\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2}-\mu\right)\Gamma\left(2\mu+\frac{1}{2}-\frac{s}{2}\right)}$ is a rational function. Note that r_μ is a constant function if and only if $\mu = 0$. The last two equations imply that $g(s) = a_1 r_\mu(4\mu+1-s)\zeta(s-2\mu)$. Both $g(s)$ and the reciprocal of $\zeta(s-2\mu)$ have Dirichlet series representations for σ sufficiently large. Hence $a_1 r_\mu(s)$ must also possess a Dirichlet series representation for σ sufficiently small. But a nonconstant rational function cannot have a Dirichlet series expansion. To see this, consider a fixed vertical line which lies in the domain of absolute convergence of the Dirichlet series. Along this line the rational function is either unbounded or has a limit, whereas the (nonconstant) Dirichlet series is bounded and does not have a limit. This latter fact can be shown by using standard mean-value formulas for Dirichlet series (it also follows immediately from the theory of almost periodic functions). So $a_1 r_\mu(s)$ is a constant and therefore either $a_1 = 0$ or $\mu = 0$.

This proves the claim. □

Remark 1. *Without resorting to the functional equation for $\zeta(s)$, we can show that there exists at most one k (where $k - \frac{1}{2}$ must be an even integer) such that the functional equation (2) has a nontrivial solution (which must be $f(s) = a_1\zeta(s - k + \frac{1}{2})$).*

Remark 2. *The functional equation (2) is rather restrictive. Indeed, M. Knopp [4] has shown that if all the hypotheses on f in Theorem 2 remain unchanged and that (2) is replaced with*

$$f(s)\Gamma(s)\pi^{-s} = f(k-s)\Gamma(k-s)\pi^{-(k-s)},$$

then this functional equation admits infinitely many linearly independent solutions.

Remark 3. *The main objective of this note is to demonstrate how Hamburger's Theorem can be sharpened by methods which were available to Siegel in the 20's. We have no doubt that similar results can be established for a wider class of functions, such as the Selberg class of zeta-functions.*

References

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