ON BINARY QUADRATIC FORMS AND THE HECKE GROUPS

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April 1, 2008

Abstract

We present a reduction theory for certain binary quadratic forms with coefficients in $\mathbb{Z}[\lambda]$, where λ is the minimal translation in a Hecke group. We generalize from the modular group $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$ to the Hecke groups and make extensive use of modified negative continued fractions. We also define and characterize "reduced" and "simple" hyperbolic fixed points of the Hecke groups.

The modular group and negative continued fractions play key roles in the reduction of binary quadratic forms with integer coefficients as presented in [12]. In this paper we generalize from the modular group to the Hecke groups by replacing $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with $\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$ for certain values of λ between 1 and 2. We use a variant of Rosen's λ-continued fractions [8], which are associated with the Hecke groups. We add to work done in this direction by Schmidt [9] and Schmidt and Sheingorn [10]. The result is a theory of reduction for a class of indefinite binary quadratic forms with coefficients in $\mathbb{Z}[\lambda]$.

Part of the motivation for this investigation is our wish to uncover properties of $\mathbb{Z}[\lambda]$-binary quadratic forms which will be useful for characterizing rational period functions of automorphic integrals on the Hecke groups [2].

In Section 1 we define the Hecke groups, λ-binary quadratic forms, and negative λ-continued fractions. In Section 2 we describe a one-to-one correspondence between hyperbolic elements of Hecke groups, certain λ-binary quadratic forms, and hyperbolic numbers. In Section 3 we characterize the "reduced" hyperbolic points of the Hecke groups. Section 4 contains the main reduction theorem. In Section 5 we characterize "simple" forms and show that they may be put into cycles corresponding to equivalence classes.

1 PRELIMINARIES

In this section we define the Hecke groups, along with the associated binary quadratic forms and continued fractions. We also prove several basic properties of the continued fractions.

1.1 HECKE GROUPS

Let $S = S_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where λ is a positive real number. Put $G(\lambda) = \langle S, T\rangle / \{\pm I\} \subseteq \mathrm{PSL}(2, \mathbb{R})$. Erich Hecke [4] showed that the only values of λ for which $G(\lambda)$ is discrete are

$$\lambda = \lambda_p = 2\cos(\pi/p),$$

for $p = 3, 4, 5, \ldots$, and $\lambda \geq 2$. We will focus on the discrete groups with $\lambda < 2$, *i.e.*, those with $\lambda = \lambda_p$, $p \geq 3$. These groups have come to be known as the *Hecke groups*, and we will denote them by $G_p = G(\lambda_p)$ for $p \geq 3$. The first several of these Hecke groups are $G_3 = G(1) = \Gamma(1)$ (the modular group), $G_4 = G(\sqrt{2})$, $G_5 = G\left(\frac{1+\sqrt{5}}{2}\right)$, and $G_6 = G(\sqrt{3})$.

Fix $p \geq 3$ and let $U = U_{\lambda_p} = S_{\lambda_p} T \in G_p$. As matrices, T and U satisfy $T^2 = U^p = -I$, but as elements of G_p they satisfy

$$T^2 = U^p = I,$$

since $M = -M$ for $M \in \mathrm{PSL}(2, \mathbb{R})$.

The entries of elements of G_p are in $\mathbb{Z}[\lambda_p]$, which for each $p \geq 3$ is the ring of algebraic integers for $\mathbb{Q}(\lambda_p)$. For $p > 3$, $\mathbb{Q}(\lambda_p)$ has nontrivial units and may have a nontrivial class group. For example, $h_{\mathbb{Q}(\lambda_{68})} = 2$. This is known to be the only class number greater than 1 for $p \leq 73$ [11].

It is well-known that $G_3 = \mathrm{PSL}(2, \mathbb{Z}[\lambda_3])$ (*i.e.*, $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$), however for the other Hecke groups $G_p \subsetneqq \mathrm{PSL}(2, \mathbb{Z}[\lambda_p])$.

An element $M \in G_p$ is *primitive* if it cannot be written as a nontrivial power of another element of the group, that is, if there exists no $V \in G_p$ such that $M = V^j$ for any integer $j > 1$.

Elements of the Hecke group act on $\mathbb{C}$ as linear fractional transformations. We say that complex numbers z_1 and z_2 are *G_p-equivalent*, and we write $z_1 \sim z_2$, if there exists $V \in G_p$ such that $Vz_1 = z_2$. This is an equivalence relation, so G_p partitions complex numbers into equivalence classes.

An element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$ is *hyperbolic* if $|a+d| > 2$, *parabolic* if $|a+d| = 2$, and *elliptic* if $|a+d| < 2$. A complex number z is a *fixed point* of $M \in G_p$ if $Mz = z$, so a nontrivial element $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ fixes

$$\begin{aligned} z &= \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c} \\ &= \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}. \end{aligned} \tag{1}$$

From this it is clear that hyperbolic elements of G_p each have two distinct real fixed points. Nontrivial parabolic elements each have one real fixed point and elliptic elements each have nonreal fixed points which are complex conjugates of each other.

Since G_p is discrete, the *stabilizer* of any complex number z in G_p, $\text{stab}(z) = \{M \in G_p \mid Mz = z\}$ is a cyclic subgroup of G_p [6, page 15]. Thus the fixed point sets of any two nontrivial elements of G_p are identical or disjoint, and all nontrivial elements of a stabilizer have identical fixed points. We designate fixed points as *hyperbolic*, *parabolic*, or *elliptic* according to whether the linear fractional transformations fixing them are hyperbolic, nontrivial parabolic, or elliptic, respectively.

If two complex numbers are G_p-equivalent, then the linear fractional transformations fixing them are conjugate to each other in G_p. Since conjugation in G_p preserves the trace, G_p-equivalent fixed points share the designation as hyperbolic points, parabolic points, or as elliptic points. As a result, equivalence classes of numbers contain either all fixed points of the same kind, or no fixed points. We designate equivalence classes containing fixed points as hyperbolic, parabolic, or elliptic.

We define the *Hecke conjugate* of any hyperbolic fixed point of G_p to be the other fixed point of the elements in its stabilizer. We denote the Hecke conjugate of α by α'. A straightforward calculation shows that if α is hyperbolic and $M \in G_p$, then $(M\alpha)' = M\alpha'$.

1.2 BINARY QUADRATIC FORMS

We consider binary quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2,$$

with coefficients in $\mathbb{Z}[\lambda_p]$. We denote such a form by $Q = [A, B, C]$ and refer to it as a λ_p-BQF or λ-BQF. We restrict our attention to indefinite forms, which have positive discriminant $D = B^2 - 4AC$.

Elements of a Hecke group act on λ-BQFs by $(Q \circ M)(x, y) = Q(ax+by, cx+dy)$ for Q a λ-BQF and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$. More explicitly, if $Q = [A, B, C]$ we have that

$$[A, B, C] \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = [A', B', C'],$$

where

$$A' = Aa^2 + Bac + Cc^2 = Q(a, c),$$

$$B' = 2Aab + B(ad + bc) + 2Ccd,$$

and

$$C' = Ab^2 + Bbd + Cd^2 = Q(b, d).$$

Note that $Q \circ M = Q \circ (-M)$, so this action does not depend on our choice of coset representative. A straightforward calculation shows that $B'^2 - 4A'C' = B^2 - 4AC$, so the action of a Hecke group preserves the discriminant.

We say that Q and Q' are *G_p-equivalent*, and write $Q \sim Q'$, if there exists a $V \in G_p$ such that $Q' = Q \circ V$. It is easy to check that G_p-equivalence is an equivalence relation, so G_p partitions the λ-BQFs into equivalence classes of forms.

1.3 CONTINUED FRACTIONS

Rosen [8] introduced a class of continued fractions closely associated with the Hecke groups. He expanded any real number into a continued fraction using a *nearest*-multiple-of-λ algorithm. We will use the *next*-multiple-of-λ algorithm [10], a modification of Rosen's continued fractions. The resulting *negative* (or *backwards*) continued fractions generalize the simple negative continued fractions in [12] for which $\lambda = 1$ and $p = 3$. Gröchenig and Haas have also studied the dynamics of these generalized backwards continued fractions in [3].

Let $r_j \in \mathbb{Z}$ for $j \geq 0$. For $p \geq 3$ we define a *finite λ_p-continued fraction* (λ_p-CF or λ-CF) by

$$\begin{aligned}[r_0; r_1, \ldots, r_n] &= r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{\ddots - \cfrac{1}{r_n\lambda}}} \\ &= \left(S^{r_0}TS^{r_1}T\ldots S^{r_n}T\right)(\infty).\end{aligned}$$

Define an *infinite* λ_p-CF by

$$[r_0; r_1, \ldots] = \lim_{n\to\infty} [r_0; r_1, \ldots, r_n],$$

if the limit exists.

We can expand any finite real number α into a unique λ-CF according to the next-multiple-of-λ algorithm. Let $\alpha_0 = \alpha$ and for $j \geq 0$ define

$$r_j = \left[\frac{\alpha_j}{\lambda}\right] + 1, \tag{2}$$

and the $j+1^{\text{st}}$ *complete quotient*

$$\alpha_{j+1} = \frac{1}{r_j\lambda - \alpha_j}. \tag{3}$$

Here $[\,\cdot\,]$ is the greatest integer function. Then $\alpha_j = r_j\lambda - \frac{1}{\alpha_{j+1}}$ for $j \geq 0$ and $[r_0; r_1, \ldots]$ is the λ-CF for α, while $[r_j; r_{j+1}, \ldots]$ is the λ-CF for α_j. We note that (3) implies that $\alpha_j \geq \frac{1}{\lambda}$ for $j \geq 1$.

We define an *admissible* λ-CF to be one that arises from a finite real number by the next-multiple-of-λ algorithm. Then we have

Lemma 1. *Fix $p \geq 3$ and put $\lambda = \lambda_p$. Then every admissible λ-CF converges.*

Proof. Put $S = S_\lambda$ and let $[r_0; r_1, \ldots]$ be the admissible λ-CF for $\alpha \in \mathbb{R}$. For any $n \geq 0$ we define $C_n = [r_0; r_1, \ldots, r_n]$, the *nth convergent* of the λ-CF. We will show that $\{C_n\}_{n=0}^{\infty}$ is decreasing and bounded below by α.

We define $C_{m,n} = [r_m; r_{m+1}, \ldots, r_n]$ for $0 \leq m \leq n$ and note that $C_{n,n} = r_n\lambda > \alpha_n$. For all $n > 0$, $\alpha_n > 0$ and $C_{n-1,n} = S^{r_{n-1}}T\left(C_{n,n}\right) > S^{r_{n-1}}T\left(\alpha_n\right) = \alpha_{n-1}$, since $S^jT(x) = j\lambda - \frac{1}{x}$ increases monotonically on $(0, +\infty)$. Continuing, we have that $C_{m,n} > \alpha_m$ for $0 \leq m \leq n$. In particular, $C_n = C_{0,n} > \alpha_0 = \alpha$ for all $n \geq 0$.

In order to show that $\{C_n\}_{n=0}^{\infty}$ is decreasing, we fix $n \geq 0$ and note that $C_{n,n} = r_n\lambda > r_n\lambda - \frac{1}{r_{n+1}\lambda} = C_{n,n+1}$. Then for $n > 0$, $C_{n-1,n} = S^{r_{n-1}}T\left(C_{n,n}\right) > S^{r_{n-1}}T\left(C_{n,n+1}\right) = C_{n-1,n+1}$. Continuing, we have that $C_{m,n} > C_{m,n+1}$ for all m, $0 \leq m \leq n$. In particular, $C_n > C_{n+1}$ for all $n \geq 0$. □

A *periodic* λ_p-CF is one for which there exist $n \geq 0$ and $m \geq 1$ such that $r_{j+m} = r_j$ for all $j \geq n$. We will take n and m to be the smallest integers for which this happens, and write a periodic λ-CF as

$$[r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}].$$

A *purely periodic* λ-CF has $n = 0$.

Schmidt and Sheingorn showed that periodic λ-CFs identify the fixed points of G_p. The proof of the following result is contained in Lemmas 1, 2, and 3 in [10].

Lemma 2. *A real number is a fixed point of G_p, $p \geq 3$, if and only if it has a periodic λ_p-CF expansion. Moreover, such a number is parabolic if and only if* *its λ_p-CF has the period* $[\overline{2, \underbrace{1, \dots, 1}_{p-3}}]$*, and is hyperbolic if and only if its λ_p-CF has a period other than* $[\overline{2, \underbrace{1, \dots, 1}_{p-3}}]$.

There are restrictions on admissible λ-CFs. We have

Lemma 3. *Put $p \geq 3$. An admissible λ_p-CF*

(i) is infinite,

(ii) has at most $p-3$ consecutive ones in any position but the beginning, and

(iii) has at most $p-2$ consecutive ones at the beginning.

Remark. In the classical case ($\lambda_3 = 1$) Lemma 3 reduces to the fact that admissible simple negative continued fractions have no ones in any position other than the first.

Proof. Set $\lambda = \lambda_p$, $S = S_\lambda$ and $U = U_\lambda$. By (2) and (3) we must have $r_j \geq 1$ for $j \geq 1$, so *(i)* must hold.

A calculation shows that

$$0 = U^p(0) < U^{p-1}(0) < \cdots < U^2(0) < U(0) = \infty,$$

so we may write the extended real line as a disjoint union of p half-open intervals,

$$[-\infty, \infty) = [-\infty, U^p(0)\,) \cup [U^p(0), U^{p-1}(0)\,) \cup \cdots \cup [U^2(0), U(0)\,). \qquad (4)$$

U maps each interval to the previous interval, the first to the last, and left endpoints to left endpoints.

Let $\alpha \in \mathbb{R}$. For any $m \geq 0$ and $j \geq 1$ we have $\alpha_m = S^{r_m} T S^{r_{m+1}} T \cdots S^{r_{m+j-1}} T \alpha_{m+j}$. If the λ-CF for α has j consecutive ones starting with r_m, then $\alpha_m = U^j \alpha_{m+j}$.

By the next multiple of λ algorithm we have that for any $n \geq 1$, $\alpha_n \in [1/\lambda, \infty)$, which is the union of the last $p-2$ intervals in (4). Thus for $m \geq 1$ we must have $j \leq p-3$. Otherwise $\alpha_{m+k} \notin [1/\lambda, \infty)$ for some k, $1 \leq k \leq j$, since U maps each interval to the previous interval in (4). □

The restrictions in Lemma 3 are the best possible, since for any $\lambda = \lambda_p$, $p \geq 3$, $\frac{3}{2}\lambda + \frac{1}{2}\sqrt{\lambda^2+4} = [\overline{3; \underbrace{1, \ldots, 1}_{p-3}}]$ and $(ST)^{-2}[\overline{3; \underbrace{1, \ldots, 1}_{p-3}}] = [1, \overline{\underbrace{1, \ldots, 1}_{p-3}, 3}]$.

2 BINARY QUADRATIC FORMS AND HYPERBOLIC NUMBERS

The reduction theory for classical binary quadratic forms in [12] uses a straightforward correspondence between primitive binary quadratic forms with integer coefficients (λ_3-BQFs) and quadratic irrational numbers. That correspondence does not directly generalize to one between λ_p-BQFs and numbers in quadratic extensions of $\mathbb{Z}[\lambda_p]$, each of which contains nontrivial units for $p > 3$. In the general setting we only define a correspondence between hyperbolic fixed points of G_p and certain λ_p-BQFs. In the process we show that these two sets are also isomorphic to the set of primitive hyperbolic linear fractional transformations (LFTs) in G_p. Because of our restriction, the ensuing reduction theory is only for λ_p-BQFs associated with hyperbolic fixed points of G_p.

Fix $p \geq 3$ and define λ_p-LFT $= \{M \in G_p | M$ is primitive, hyperbolic$\}$ and λ_p-BQF $= \{Q | Q$ is a λ_p-BQF$\}$. We note that $M \in \lambda_p$-LFT if and only if $M^{-1} \in$

λ_p-LFT, since inverses of primitive, hyperbolic linear fractional transformations are primitive and hyperbolic. Define $\rho : \lambda_p\text{-LFT} \to \lambda_p\text{-BQF}$ by $\rho(M) = Q_M = [c, d-a, -b]$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with positive trace. (We select elements with positive trace so that ρ is well-defined, since $M = -M$ in $\mathrm{PSL}(2,\mathbb{R})$, and $-M$ would map to $-Q_M$.)

The coefficients of Q_M are those in the quadratic equation that gives the fixed point formula (1). Although $M^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ has the same fixed points as M, we have $\rho\left(M^{-1}\right) = Q_{M^{-1}} = [-c, a-d, b] = -Q_M$.

Elements in the range of ρ are indefinite λ-BQFs, since the discriminant of Q_M is $D = (a+d)^2 - 4$ and elements of the domain are hyperbolic. Since binary quadratic forms in the range are images of hyperbolic matrices, we call those forms *hyperbolic* λ-BQFs and denote the range of ρ by λ_p-HBQF.

Define $\sigma : \lambda_p\text{-HBQF} \to \mathbb{R}$ by $\sigma(Q) = \alpha_Q = \frac{-B+\sqrt{D}}{2A}$ for $Q = [A, B, C]$ and $D = B^2 - 4AC$. The real number α_Q is one of the zeros of $Q(x, 1)$ and lies in $\mathbb{Q}(\lambda_p, \sqrt{D})$. The other zero of $Q(x, 1)$ is the image of $-Q$. This definition of σ is a (slight) generalization of an association of binary quadratic forms to numbers which Schmidt and Sheingorn use in [10] (and Zagier uses for the classical case in [12]), where they require that $A > 0$.

We calculate that $(\sigma \circ \rho)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \sigma[c, d-a, -b] = \frac{a-d+\sqrt{D}}{2c}$, where $D = (a+d)^2 - 4$. Thus $\sigma \circ \rho$ maps every primitive hyperbolic element $M \in G_p$ to one of its fixed points. Another calculation shows that if $(\sigma \circ \rho)(M) = \alpha$, then $(\sigma \circ \rho)\left(M^{-1}\right) = \alpha'$, so Hecke conjugates are images of inverses in λ_p-LFT (and of negative forms in λ_p-HBQF). The following lemma shows that $\sigma \circ \rho$ maps to attracting fixed points.

Lemma 4. *Fix $p \geq 3$. If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_p$ with $a + d > 2$, then $\alpha = \frac{a-d+\sqrt{D}}{2c}$ is the attracting fixed point of M.*

Remark. Note that for this lemma we must select a coset representative with positive trace, and that this is always possible.

Proof. We first observe that $M(\infty)$ is not between α and α'. Then $\frac{|M(\infty)-\alpha|}{|M(\infty)-\alpha'|} = \frac{\frac{a}{c} - \frac{a-d+\sqrt{D}}{2c}}{\frac{a}{c} - \frac{a-d-\sqrt{D}}{2c}} = \frac{a+d-\sqrt{D}}{a+d+\sqrt{D}} < 1$, since $a + d > 0$. Thus $M(\infty)$ is closer to α than to α', so α is attracting. □

Let λ_p-HFP denote the set of hyperbolic fixed points of G_p. Then we have the following Lemma.

Lemma 5. *Fix $p \geq 3$. The sets λ_p-LFT, λ_p-HBQF, and λ_p-HFP are isomorphic.*

Proof. We have that $(\sigma \circ \rho) : \lambda_p\text{-LFT} \to \lambda_p\text{-HFP}$. We will show that this map is an isomorphism. Then, since σ maps λ_p-LFT onto λ_p-HBQF, σ must also be an isomorphism.

To show that $\sigma \circ \rho$ is injective, we suppose that $(\sigma \circ \rho)(V) = (\sigma \circ \rho)(W) = \alpha$ for $V, W \in \lambda_p$-LFT. Then both V and W fix α, so both V and W are in $\text{stab}(\alpha)$. Since $\text{stab}(\alpha)$ is cyclic and both V and W are primitive, we have that $V = W^{-1}$ or $V = W$. If $V = W^{-1}$, then $(\sigma \circ \rho)(V) = ((\sigma \circ \rho)(W))' \neq (\sigma \circ \rho)(W)$, a contradiction. Thus $V = W$, so $\sigma \circ \rho$ is injective.

To show that $\sigma \circ \rho$ is surjective, we suppose that $\alpha \in \lambda_p$-HFP. Then $\text{stab}(\alpha) = \langle M \rangle$ for some hyperbolic $M \in G_p$. This M must be primitive (or else it does not generate stab(α)), so $M \in \lambda_p$-LFT. And if $M \in \lambda_p$-LFT, so is M^{-1}. Now M has two fixed points, α and α', so $(\sigma \circ \rho)(M) = \alpha$, or $(\sigma \circ \rho)(M) = \alpha'$. If $(\sigma \circ \rho)(M) = \alpha'$, then $(\sigma \circ \rho)\left(M^{-1}\right) = \alpha$. In either case α is the image of an element in λ_p-LFT. □

Since $(\sigma \circ \rho) : \lambda_p\text{-LFT} \to \lambda_p\text{-HFP}$ is an isomorphism, so is $\tau = (\sigma \circ \rho)^{-1} : \lambda_p\text{-HFP} \to \lambda_p\text{-LFT}$. We will use λ-CFs to write a formula for τ.

Lemma 6. *Fix $p \geq 3$ and suppose that $\alpha = [r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}] \in \lambda_p$-HFP. The corresponding linear fractional transformation is given by $\tau(\alpha) = VWV^{-1}$, where $V = S^{r_0}TS^{r_1}T \cdots S^{r_{n-1}}T$ and $W = S^{r_n}TS^{r_{n+1}}T \cdots S^{r_{n+m-1}}T$.*

Proof. We have that VWV^{-1} is primitive since every λ_p-CF period is minimal. We also have that VWV^{-1} is hyperbolic, since it fixes α. Thus $VWV^{-1} \in \lambda_p$-LFT and fixes α, so $(\sigma \circ \rho)\left(VWV^{-1}\right) = \alpha$ or $(\sigma \circ \rho)\left(VWV^{-1}\right) = \alpha'$. By Lemma 4 our proof will be complete if we show that α is the attracting fixed point of VWV^{-1}, which means that $(\sigma \circ \rho)\left(VWV^{-1}\right) = \alpha$, or equivalently $\tau(\alpha) = VWV^{-1}$.

In order to show that α is the attracting fixed point of VWV^{-1}, we consider images of $x = [r_0; r_1, \ldots, r_{n-1}]$ under applications of VWV^{-1}. Note that $[r_0; r_1, \ldots, r_{n-1}]$ is not an admissible λ-CF; technically we have that $x = [r_0; r_1, \ldots, r_{n-1}+1, \overline{\underbrace{1, \ldots, 1}_{p-3}, 2}]$. But $VWV^{-1}x = [r_0; r_1, \ldots, r_{n-1}, r_n, \ldots, r_{n+m-1}+1, \overline{\underbrace{1, \ldots, 1}_{p-3}, 2}]$, and $\left(VWV^{-1}\right)^k x \to \alpha$ as $k \to \infty$, so VWV^{-1} attracts to α. □

The following lemma allows us to translate between the action of G_p on λ-BQFs and the action of G_p on hyperbolic numbers.

Lemma 7. *Fix $p \geq 3$ and let $\lambda = \lambda_p$. Suppose that Q_α and Q_β are hyperbolic λ-BQFs associated with hyperbolic numbers α and β, respectively, and let $V \in G_p$. Then $Q_\beta = Q_\alpha \circ V$ if and only if $\beta = V^{-1}\alpha$.*

Proof. We write $Q_\alpha = [A, B, C]$, so $\alpha = \frac{-B+\sqrt{D}}{2A}$ where $D = B^2 - 4AC$, and we write $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Suppose that $Q_\beta = Q_\alpha \circ V$. Then

$$\begin{aligned} Q_\beta &= [A, B, C] \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\ &= [Aa^2 + Bac + Cc^2, 2Aab + B(ad + bc) + 2Ccd, Ab^2 + Bbd + Cd^2], \end{aligned}$$

so

$$\beta = \frac{-(2Aab + B(ad + bc) + 2Ccd) + \sqrt{D}}{2(Aa^2 + Bac + Cc^2)}.$$

On the other hand, $V^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, so

$$\begin{aligned} V^{-1}\alpha &= \frac{d\left(\frac{-B+\sqrt{D}}{2A}\right) - b}{-c\left(\frac{-B+\sqrt{D}}{2A}\right) + a} \\ &= \frac{-2Ab - Bd + c\sqrt{D}}{2Aa - Bc - c\sqrt{D}} \\ &= \frac{-(2Aab + B(ad + bc) + 2Ccd) + \sqrt{D}}{2(Aa^2 + Bac + Cc^2)} \\ &= \beta. \end{aligned}$$

Next we suppose that $\beta = V^{-1}\alpha$ and write $\tau(\alpha) = M_\alpha = \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right) \in \lambda_p$-LFT. Then $\rho\,(M_\alpha) = Q_\alpha = [t, u - r, -s]$, so

$$\begin{aligned} Q_\alpha \circ V &= [t, u - r, -s] \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\ &= [ta^2 + (u - r)ac - sc^2, 2tab + (u - r)(ad + bc) + 2scd, tb^2 + (u - r)bd - sd^2]. \end{aligned}$$

Consider the linear fractional transformation $V^{-1}M_\alpha V$, which is hyperbolic since trace is invariant under conjugation, and primitive since M_β is primitive. Thus $V^{-1}M_\alpha V \in \lambda_p$-LFT. Now β is the attracting fixed point for $V^{-1}M_\alpha V$, since $V\beta = \alpha$ is the attracting fixed point for M_α. Thus $V^{-1}M_\alpha V = M_\beta = \tau(\beta)$. Now

$$\begin{aligned} M_\beta &= V^{-1}M_\alpha V \\ &= \begin{pmatrix} adr + cds - abt - bcu & -tb^2 + (r-u)bd + sd^2 \\ ta^2 + (u-r)ac - sc^2 & -cbr - cds + abt + adu \end{pmatrix}, \end{aligned}$$

so

$$\begin{aligned} Q_\beta &= \rho\left(M_\beta\right) \\ &= \left[ta^2 + (u-r)ac - sc^2, 2tab + (u-r)(ad+bc), tb^2 + (u-r)bd - sd^2\right] \\ &= Q_\alpha \circ V. \end{aligned}$$

□

By Lemma 7, G_p-equivalence of hyperbolic λ_p-BQFs corresponds to G_p-equivalence of hyperbolic numbers. Thus every G_p-equivalence class of λ_p-BQFs contains either only hyperbolic forms or no hyperbolic forms, so we designate λ_p-BQF equivalence classes themselves as hyperbolic or non-hyperbolic.

3 REDUCED NUMBERS

Fix $p \geq 3$. We say that a real number α is a G_p-*reduced* number if the λ_p-CF expansion of α is purely periodic with period other than $[\overline{2, \underbrace{1, \ldots, 1}_{p-3}}]$. If α is G_p-reduced it is hyperbolic, and we say that the associated hyperbolic λ_p-BQF Q_α is G_p-*reduced.*

The following Lemma is a modification of a familiar result from classical continued fractions. It is stated without proof in [10, page 389].

Lemma 8. *Let $p \geq 3$ and put $\lambda = \lambda_p$. If $\alpha = [\overline{r_0; r_1, \ldots, r_n}]$ is a λ-CF, then* $\frac{1}{\alpha'} = [\overline{r_n; r_{n-1}, \ldots, r_0}]$.

Proof. Put $S = S_\lambda$. We have $\alpha_j = [\overline{r_j; r_{j+1}, \ldots}]$ for $j \geq 0$. Then $\alpha_{j+1} = TS^{-r_j}\alpha_j$, and $\alpha_0 = \alpha_{n+1} = TS^{-r_n}\alpha_n$. Taking Hecke conjugates, we have $\alpha'_{j+1} = TS^{-r_j}\alpha'_j = \frac{1}{r_j\lambda - \alpha'_j}$, $j \geq 0$, and $\alpha'_0 = \frac{1}{r_n\lambda - \alpha'_n}$, so $\frac{1}{\alpha'_0} = r_n\lambda - \alpha'_n$. We combine these to get

$$\begin{aligned} \frac{1}{\alpha'_0} &= r_n\lambda - \cfrac{1}{r_{n-1}\lambda - \cfrac{1}{r_{n-2}\lambda - \ddots}} \\ &= [\overline{r_n; r_{n-1}, \ldots, r_0}]. \end{aligned}$$

□

We need the following result for the proof of the next Theorem.

Lemma 9. *Fix $p \geq 3$ and let $U = U_{\lambda_p}$. Then $\frac{1}{U^k(0)} = U^{p-k+1}(0)$ for any integer k.*

Proof. Let $c_k = \frac{\sin(k\pi/p)}{\sin(\pi/p)}$ for $k \geq 0$. Meier and Rosenberger [7] show that as linear fractional transformations

$$U^k = \left(\begin{smallmatrix} c_{k+1} & -c_k \\ c_k & -c_{k-1} \end{smallmatrix}\right), \tag{5}$$

for $k \in \mathbb{Z}^+$. In fact, it is easy to show that (5) holds for all integers k. Then

$$\begin{aligned} U^{p-k+1}(0) &= \frac{c_{p-k+1}}{c_{p-k}} \\ &= \frac{\sin((p-k+1)\pi/p)}{\sin((p-k)\pi/p)} \\ &= \frac{\sin((k-1)\pi/p)}{\sin(k\pi/p)} \\ &= \frac{c_{k-1}}{c_k} \\ &= \frac{1}{U^k(0)}. \end{aligned}$$

□

We next characterize G_p-reduced numbers.

Theorem 1. *Put $p \geq 3$, $\lambda = \lambda_p$, and $U = U_\lambda$. Suppose that α is a hyperbolic fixed point of G_p. Then α is G_p-reduced with k leading ones in its λ-CF if and only if*

$$0 < \alpha' < U^{k+2}(0) < \alpha < U^{k+1}(0), \tag{6}$$

for some integer k, $0 \leq k \leq p-3$. Furthermore, the λ-CF for α has j trailing ones in its period if and only if

$$U^{p-j}(0) < \alpha' < U^{p-j-1}(0),$$

for some integer j, $0 \leq j \leq p-k-3$.

Remark. In the classical case ($\lambda_3 = 1$) we must have that $k = 0$. Then a hyperbolic point α is G_3-reduced if and only if $0 < \alpha' < 1 < \alpha < \infty$. This agrees with the definition of a reduced number in the classical case [12, page 128].

Proof. We first suppose that α is G_p-reduced and that the λ-CF for α has k leading ones. Then $0 \le k \le p-3$. We may write $\alpha = [\overline{r_0; r_1, \dots, r_{m-1}}] = [\overline{\underbrace{1;1,\dots,1}_{k}, r_k, r_{k+1}, \dots, r_{m-1}}]$, and $\alpha_j = [\overline{r_j; r_{j+1}, \dots, r_{m-1}, r_0, \dots, r_{j-1}}]$ for $0 \le j \le m-1$. Each α_j is hyperbolic and thus not a multiple of λ. Since $r_k \ge 2$ we have $\alpha_k > \lambda = U^2(0)$, *i.e.*, α_k is in the pth interval of (4). Now $r_j = 1$ for $0 \le j < k$ implies that $\alpha = U^k \alpha_k$, so α is in the $(p-k)$th interval of (4), *i.e.*, $U^{k+2}(0) < \alpha < U^{k+1}(0)$. By Lemma 8 we have $\frac{1}{\alpha'} = [\overline{r_{m-1}; \dots, r_{k+1}, r_k, \underbrace{1, \dots, 1}_{k}}] > 0$, so $\alpha' > 0$. Also, $\alpha_k = [\overline{r_k; r_{k+1}, \dots, r_{m-1}, \underbrace{1, \dots, 1}_{k}}]$, so $\frac{1}{\alpha'_k} = [\overline{\underbrace{1;1,\dots,1}_{k}, r_{m-1}, \dots, r_{k+1}, r_k}] > \frac{1}{\lambda} = U^{p-1}(0)$. Then $\frac{1}{\alpha'} = U^{-k}\left(\frac{1}{\alpha'_k}\right) > U^{p-k-1}(0) > 0$, since U^{-k} maps every interval in (4) k intervals to the right, and $2 \le p-k-1 \le p-1$. Thus $\alpha' < \frac{1}{U^{p-k-1}(0)} = U^{k+2}(0)$, by Lemma 9, and we have verified (6).

Next, we suppose that α is a hyperbolic fixed point of G_p that satisfies (6) for some integer k, $0 \le k \le p-3$. By Lemma 2 we may write $\alpha = [r_0; r_1, \dots, r_{n-1}, \overline{r_n, \dots, r_{n+m-1}}]$. If any complete quotient α_j satisfies $0 < \alpha_j < \lambda$, then $r_j = 1$ and α_j is in one of the second through $(p-1)$st intervals of (4). In this case $\alpha_{j+1} = U^{-1}\alpha_j$, so α_{j+1} is in the interval to the right of α_j. Taking Hecke conjugates we have $\alpha'_{j+1} = U^{-1}\alpha'_j$, so α'_{j+1} is in the interval to the right of α'_j. If α satisfies (6) with $k=0$ then $\alpha > \lambda$, so $r_0 \ge 2$ and the λ-CF for α has no leading ones. If α satisfies (6) with $k > 0$, then $0 < \alpha < U^{k+1}(0) \le U^2(0) = \lambda$, so $r_0 = 1$. Then since $\alpha_1 = U^{-1}\alpha$ and $\alpha'_1 = U^{-1}\alpha'$, we have that $0 < \alpha'_1 < U^{k+1}(0) < \alpha_1 < U^k(0)$. We repeat this argument, and for $0 \leq j \leq k-1$ we have $0 < \alpha'_j < U^{k+2-j}(0) < \alpha_j < U^{k+1-j}(0) \leq \lambda$, so $r_j = 1$, $\alpha_{j+1} = U^{-1}\alpha_j$, and $\alpha'_{j+1} = U^{-1}\alpha'_j$. For $j = k$ we have $0 < \alpha'_k < U^2(0) < \alpha_k < U(0)$, so $\alpha_k > \lambda$ and $r_k \geq 2$. Thus the λ-CF for α has k leading ones.

In order to show that α is G_p-reduced, we first claim that $0 < \alpha'_j < \lambda$ for all $j \geq 0$. Since α' is in one of the second through $(p-k-1)$st intervals of (4), and the λ-CF for α has k leading ones, the discussion above implies that each α'_j, $0 \leq j \leq k$ is in one of the second through $(p-1)$st intervals, so $0 < \alpha'_j < \lambda$ for $0 \leq j \leq k$. A calculation shows that if $0 < \alpha'_t < \lambda$ and $r_t \geq 2$ for any t, then $0 < \alpha'_{t+1} < \frac{1}{\lambda}$ and α'_{t+1} is in the second interval of (4). If r_t is followed by $r_{t+1} \geq 2$, then $0 < \alpha'_{t+2} < \frac{1}{\lambda}$ by the same calculation. On the other hand, if r_t is followed by ℓ ones, α'_j is in one of the second through $(\ell+2)$nd intervals for $t+1 \leq j \leq t+\ell+1$. Since the λ-CF for α'_j is admissible, $\ell \leq p-3$, so each such α'_j is in one of the second through $(p-1)$st intervals. Thus $0 < \alpha'_j < \lambda$ for the ℓ complete quotients following α_t. The claim follows by induction on j. We next note that for every $j \geq 1$, the complete quotient α_j has a unique predecessor α_{j-1} with $0 < \alpha'_j < \lambda$. Indeed, $0 < \alpha'_j < \lambda$ implies that $T\alpha'_j < 0$, so $0 < S^tT\alpha'_j < \lambda$ for a unique t, $t \geq 1$. But since $0 < \alpha'_{j-1} < \lambda$ and $\alpha'_{j-1} = S^{r_{j-1}}T\alpha'_j$, we must have that $r_{j-1} = t$ is uniquely determined. Then $\alpha_n = \alpha_{n+m}$ implies that $\alpha_j = \alpha_{j+m}$ for every $j < n$ and $\alpha = [\overline{r_0; r_1, \ldots, r_{m-1}}]$. Thus α is reduced, and we have proved the first part of the theorem.

To prove the second part of the theorem we suppose the λ-CF for α has j

trailing ones in its period. Then the λ-CF for α has $j+k$ consecutive ones, so $0 \leq j \leq p-k-3$. Now $\alpha = [\overline{r_0; r_1, \ldots, r_{m-j-1}, \underbrace{1, \ldots, 1}_{j}}]$, with $r_{m-j-1} \geq 2$, and $\frac{1}{\alpha'} = [\overline{\underbrace{1; \ldots, 1}_{j}, r_{m-j-1}, \ldots, r_1, r_0}]$ by Lemma 8. The first part of the theorem implies that $U^{j+2}(0) < \frac{1}{\alpha'} < U^{j+1}(0)$, so $\frac{1}{U^{j+1}(0)} < \alpha' < \frac{1}{U^{j+2}(0)}$. An application of Lemma 9 gives $U^{p-j}(0) < \alpha' < U^{p-j-1}(0)$.

Finally, we suppose that $U^{p-j}(0) < \alpha' < U^{p-j-1}(0)$ for some integer j, $0 \leq j \leq p-k-3$. We can reverse every step in the argument above to conclude that the λ-CF for α has j trailing ones in its period, and we have proved the second part of the theorem. □

4 REDUCTION

In this section we generalize the reduction theory of indefinite binary quadratic forms to hyperbolic λ-BQFs. We see that every hyperbolic G_p-equivalence class of forms contains a unique cycle of reduced forms. From this it follows that there are a finite number of hyperbolic equivalence classes for each discriminant.

The following theorem generalizes Satz 1 of [12, page 122].

Theorem 2. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Every hyperbolic λ-BQF Q may be transformed into a reduced λ-BQF by finitely many applications of $S_\lambda^r T$, where at each step $r = \left[\frac{\alpha_Q}{\lambda}\right] + 1$ and α_Q is the hyperbolic fixed point associated with Q. Furthermore, $S_\lambda^r T$ maps reduced forms to reduced forms, so the reduced forms fall into disjoint cycles. Finally, every equivalence between reduced λ-*

BQFs is obtained by iteration of $S_\lambda^r T$; thus each equivalence class of λ-BQFs with hyperbolic forms contains one cycle of reduced forms.

Proof. Suppose that Q_0 is a hyperbolic λ-BQF. Let $\alpha_0 = \sigma(Q_0)$ be the hyperbolic fixed point associated with Q_0. Then α_0 has a λ-CF expansion of the form

$$\alpha_0 = [r_0; r_1, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}],$$

where $r_0 = \left[\frac{\alpha_0}{\lambda}\right] + 1$. Put

$$\begin{aligned} \alpha_1 &= TS^{-r_0}\alpha_0 \\ &= [r_1; r_2, \ldots, r_{n-1}, \overline{r_n, \ldots, r_{n+m-1}}], \end{aligned}$$

with $S = S_\lambda$. We repeat this process, at each step using the mapping TS^{-r_j}, $r_j = \left[\frac{\alpha_j}{\lambda}\right] + 1$ to calculate α_{j+1}. After n steps, we have

$$\begin{aligned} \alpha_n &= TS^{-r_{n-1}} \cdots TS^{-r_0}\alpha_0 \\ &= [\overline{r_n; \ldots, r_{n+m-1}}], \end{aligned}$$

which is reduced. For $1 \leq j \leq n$ let $Q_j = \sigma^{-1}(\alpha_j)$ be the hyperbolic λ-BQF associated with the hyperbolic number α_j. Then Q_n is reduced and by Lemma 7, $Q_n = Q_0 \circ \left(S^{r_0}T \cdots S^{r_{n-1}T}\right)$.

Next, suppose that R_0 is a reduced λ-BQF. Let $\beta_0 = \sigma(R_0)$ be the reduced fixed point associated with R_0. Since β_0 is reduced, it has a λ-CF expansion of the form

$$\beta_0 = [\overline{r_0; r_1, \ldots, r_m}],$$

where $r_0 = \left[\frac{\beta_0}{\lambda}\right] + 1$. Then

$$\begin{aligned}\beta_1 &= TS^{-r_0}\beta_0 \\ &= [\overline{r_1; r_2, \ldots, r_m, r_0}],\end{aligned}$$

is also reduced. We repeat this process by putting

$$\begin{aligned}\beta_{j+1} &= TS^{-r_j}\beta_j \\ &= [\overline{r_{j+1}; \ldots, r_j}],\end{aligned}$$

where each $r_j = \left[\frac{\beta_j}{\lambda}\right] + 1$, for $1 \le j \le m-1$. We also note that

$$\beta_0 = TS^{-r_m}\beta_m,$$

where $r_m = \left[\frac{\beta_m}{\lambda}\right] + 1$, so we have a cycle of reduced numbers. For $1 \le j \le m$ let $R_j = \sigma^{-1}(\beta_j)$ be the reduced λ-BQFs associated with the reduced numbers β_j. Then the R_j, $0 \le j \le m$, form a cycle of $m+1$ reduced λ-BQFs.

Finally, suppose that Q_1 and Q_2 are reduced λ-BQFs with $Q_1 \sim Q_2$. Let $\alpha_1 = \sigma(Q_1)$ and $\alpha_2 = \sigma(Q_2)$ be the corresponding reduced numbers. Then $\alpha_1 \sim \alpha_2$, so there exists a $V \in G_p$ such that $V\alpha_1 = \alpha_2$. Since α_1 and α_2 both have purely periodic λ-CF expressions, their periods must be cyclic permutations of each other. Repeated application of TS^{-r}, $r = \left[\frac{\beta}{\lambda}\right] + 1$, as above, maps α_1 to α_2. Translating back to reduced λ-BQFs, we have that repeated application of S^rT, $r = \left[\frac{\beta_R}{\lambda}\right] + 1$, maps Q_1 to Q_2.

□

Example 1. Put $\lambda = \lambda_5 = \frac{1+\sqrt{5}}{2}$, $S = S_\lambda$, and let $\beta_0 = [2; 3, \overline{2, 1, 1, 4}]$. Then $M_0 = \left(\begin{smallmatrix} -9\lambda-6 & 51\lambda+32 \\ -3\lambda-2 & 18\lambda+9 \end{smallmatrix}\right)$ generates the stabilizer of β_0, and $Q_0 = [-3\lambda-2, 27\lambda+$

$15, -51\lambda - 32]$ is the λ_5-BQF corresponding to β_0. Now Q_0 is in a hyperbolic equivalence class $\mathcal{A}$ of forms of discriminant $D = 135\lambda + 86 = \frac{307+135\sqrt{5}}{2}$. We reduce Q_0 by

$$\begin{aligned} Q_1 &= Q_0 \circ S^2T = [\lambda + 2, -7\lambda - 3, -3\lambda - 2], \\ Q_2 &= Q_1 \circ S^3T = [3\lambda + 4, -11\lambda - 3, \lambda + 2], \end{aligned}$$

which is reduced. The cycle of reduced λ_5-BQFs in $\mathcal{A}$ is

$$\begin{aligned} Q_3 &= Q_2 \circ S^2T = [13\lambda + 8, -17\lambda - 9, 3\lambda + 4], \\ Q_4 &= Q_3 \circ ST = [11\lambda + 8, -25\lambda - 17, 13\lambda + 8], \\ Q_5 &= Q_4 \circ ST = [\lambda + 2, -13\lambda - 5, 11\lambda + 8], \text{and} \\ Q_2 &= Q_5 \circ S^4T = [3\lambda + 4, -11\lambda - 3, \lambda + 2]. \end{aligned}$$

The successive values of r in S^rT are the successive entries in the λ_5-CF for β_0. The corresponding reduced numbers are

$$\begin{aligned} \beta_2 &= TS^{-3}\beta_1 = TS^{-3}TS^{-2}\beta_0 = [\overline{2; 1, 1, 4}], \\ \beta_3 &= TS^{-2}\beta_2 = [\overline{1; 1, 4, 2}], \\ \beta_4 &= TS^{-1}\beta_3 = [\overline{1; 4, 2, 1}], \text{and} \\ \beta_5 &= TS^{-1}\beta_4 = [\overline{4; 2, 1, 1}]. \end{aligned}$$

5 SIMPLE FORMS AND NUMBERS

In this section we define simple λ-BQFs and simple numbers, which are easily characterized and are related to reduced forms and numbers. We put the simple forms and numbers into cycles which can be used to describe rational period functions.

Fix $p \geq 3$. We call a hyperbolic λ_p-BQF $Q = [A, B, C]$ G_p-*simple* if $A > 0 > C$. If Q is a simple λ-BQF, we say that the associated hyperbolic point α_Q is a G_p-*simple* number.

Lemma 10. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Suppose that $Q = [A, B, C]$ is a hyperbolic λ-BQF associated with $\alpha = \alpha_Q$. Then Q is simple if and only if $\alpha' < 0 < \alpha$.*

Proof. The proof is an exercise in calculating with inequalities. □

Next we establish the connection between reduced and simple λ_p-BQFs.

Theorem 3. *Let $p \geq 3$ and put $\lambda = \lambda_p$. Every G_p-simple λ-BQF Q is transformed into a G_p-reduced form by a single application of S_λ^{-n}, with $n = -\left[\frac{\alpha_Q'}{\lambda}\right]$. The set of G_p-simple λ-BQFs is given by*

$$\left\{ Q \circ S_\lambda^i \mid Q \text{ is } G_p\text{-reduced, } 1 \leq i \leq \left[\left[\frac{\beta_Q}{\lambda}\right]\right] \right\}.$$

Proof. Put $S = S_\lambda$ and let Q be a G_p-simple λ-BQF. Let $\alpha = \alpha_Q$ be the G_p-simple number associated with Q. Then $\alpha' < 0 < \alpha$, so $n = -\left[\frac{\alpha'}{\lambda}\right] \geq 1$ and $S^n\alpha > \lambda$. Also, $-n < \frac{\alpha'}{\lambda} < 1 - n$ implies that $0 < S^n\alpha' < \lambda$. Thus $0 < S^n\alpha' < \lambda < S^n\alpha$, and $\beta = S^n\alpha$ is a reduced number by Theorem 1.

To prove the second statement, we note that the set of hyperbolic numbers corresponding to the set of λ_p-BQFs in the statement of the theorem is

$$\mathcal{Z} = \left\{ S_\lambda^{-i}\beta \mid \beta \text{ is } G_p\text{--reduced, } 1 \leq i \leq \left[\left[\frac{\beta}{\lambda}\right]\right] \right\}.$$

We need to show that $\mathcal{Z}$ is the set of G_p-simple numbers.

We first suppose that α is any G_p-simple number, so $\alpha' < 0 < \alpha$. Then $\beta = S^i\alpha$, $i = -\left[\frac{\alpha'}{\lambda}\right]$, is reduced and $\left[\frac{\beta}{\lambda}\right] = \left[\frac{\alpha}{\lambda}\right] + i \geq i$. Now $\left[\frac{\alpha'}{\lambda}\right] \leq -1$, so $i \geq 1$, and $\alpha \in \mathcal{Z}$.

Next we suppose that $\alpha \in \mathcal{Z}$. Then $\alpha = S^{-i}\beta$, $1 \leq i \leq \left[\frac{\beta}{\lambda}\right]$, for some reduced β. Now $i \leq \left[\frac{\beta}{\lambda}\right]$ implies that $\alpha = S^{-i}\beta > 0$. The fact that $i \geq 1$, along with $\beta' < \lambda$, implies that $\alpha' = S^{-i}\beta' < 0$. Thus $\alpha' < 0 < \alpha$ and α is a G_p-simple number. $\square$

Since there are finitely many reduced λ-BQFs of a given discriminant, Theorem 3 implies that there are also finitely many simple forms of a given discriminant. Also, every reduced form Q (or number α) is connected by S^j to $\left[\frac{\beta_Q}{\lambda}\right]$ simple forms (numbers). In particular, if $\beta_Q < \lambda$ then $Q \circ S^j$ fails to be simple for any j.

Simple numbers (and associated forms) may be put into cycles, using a function defined by Schmidt [9, page 234]. We use these cycles in [2] to describe the poles of rational period functions for automorphic integrals on the Hecke groups. This function generalizes the one used by Choie and Zagier to characterize rational period functions for modular integrals [1, page 91]. For a fixed

$p \geq 3$ define $\Phi_p : [0, \infty) \to [0, \infty)$ by

$$\Phi_p(x) = \begin{cases} TUx, & U^p(0) \leq x < U^{p-1}(0) \\ TU^2x, & U^{p-1}(0) \leq x < U^{p-2}(0) \\ \vdots & \\ TU^{p-1}x, & U^2(0) \leq x, \end{cases}$$

where $U = U_{\lambda_p}$. Given $\alpha \in [0, \infty)$, the exponent i in $\Phi_p(\alpha) = TU^i\alpha$ is the unique exponent between 1 and $p-1$ for which $TU^i\alpha \in [0, \infty)$.

The following theorem generalizes a lemma for the modular group in [1].

Theorem 4. *Let $p \geq 3$ and put $\lambda = \lambda_p$. The finite orbits of Φ_p are the set $\{0\}$ and the sets*

$$\mathcal{Z}_{\mathcal{A}} = \{\alpha_Q \mid Q \in \mathcal{A}, \textit{ simple}\},$$

where $\mathcal{A}$ runs over all hyperbolic G_p-equivalence classes of λ-BQFs.

Proof. Put $S = S_\lambda$ and $U = U_\lambda$. Any finite orbit except $\{0\}$ has the form

$$\mathcal{C} = \{\alpha_1, \alpha_1 - \lambda, \ldots, \alpha_1 - m_1\lambda, \alpha_2, \alpha_2 - \lambda, \ldots, \\ \alpha_2 - m_2\lambda, \ldots, \alpha_s, \alpha_s - \lambda, \ldots, \alpha_s - m_s\lambda\},$$

for some positive real numbers $\alpha_1, \alpha_2, \ldots, \alpha_s$, where $m_j = \left[\frac{\alpha_j}{\lambda}\right] \geq 0$, $1 \leq j \leq s$. We must also have that

$$\alpha_{j+1} = TU^{i_j}(\alpha_j - m_j\lambda),$$

$1 \leq j \leq s-1$, as well as

$$\alpha_1 = TU^{i_s}(\alpha_s - m_s\lambda),$$

with $1 \le i_j \le p-2$ for all j. Now since $m_j = \left[\frac{\alpha_j}{\lambda}\right]$, we have $0 < \alpha_j - m_j\lambda < \lambda$, so $U^{\ell_j+1}(0) < \alpha_j - m_j\lambda < U^{\ell_j}(0)$ for some ℓ_j, $2 \le \ell_j \le p-1$. We have eliminated the possibility that $\alpha_j - m_j\lambda = U^{\ell_j+1}(0)$, since then $\alpha_j - m_j\lambda$ would not be part of a finite orbit of Φ_p. Thus

$$\begin{aligned} \alpha_{j+1} &= \Phi_p(\alpha_j - m_j\lambda) \\ &= TU^{p-\ell_j}S^{-m_j}\alpha_j, \end{aligned}$$

$1 \le j \le s-1$, and

$$\begin{aligned} \alpha_1 &= \Phi_p(\alpha_s - m_s\lambda) \\ &= TU^{p-\ell_s}S^{-m_s}\alpha_s. \end{aligned}$$

For each j we put $\beta_j = S\alpha_j$ and $r_j = \left[\frac{\beta_j}{\lambda}\right] + 1 = m_j + 2 \ge 2$. Then

$$\begin{aligned} \beta_{j+1} &= S\alpha_{j+1} \\ &= U^{p-\ell_j+1}S^{-m_j}\alpha_j \\ &= U^{p-\ell_j+1}S^{-m_j-1}\beta_j, \end{aligned}$$

for $j \ne s$, and

$$\beta_0 = U^{p-\ell_s+1}S^{-m_s-1}\beta_s.$$

Reversing direction and using $m_j = r_j - 2$, we have

$$\beta_j = S^{r_j}TU^{\ell_j-2}\beta_{j+1},$$

$j \ne s$, and

$$\beta_s = S^{r_s}TU^{\ell_s-2}\beta_0.$$

Therefore

$$\beta_j = [\overline{r_j; \underbrace{1, \ldots, 1}_{\ell_j - 2}, r_{j+1}, \ldots}],$$

$j \neq s$, and

$$\beta_s = [\overline{r_s; \underbrace{1, \ldots, 1}_{\ell_s - 2}, r_0, \ldots}].$$

Now $0 \leq \ell_j - 2 \leq p - 3$, so the λ-CF expansions are all admissible and β_j is G_p-reduced for each j. Clearly, the β_j are all part of the same cycle of reduced numbers in a G_p-equivalence class of numbers. Let $\mathcal{A}$ represent the equivalence class of λ-BQFs containing the corresponding forms. If any ones occur in the λ-CF expansions, the cycle contains reduced numbers other than the β_j. But these other reduced numbers are of the form $\beta = [\overline{1; \ldots}] < \lambda$, and are not connected with any G_p-simple numbers. Thus all of the simple numbers associated with forms in $\mathcal{A}$ are of the form $S^{-i}\beta_j$, $1 \leq j \leq s$, where i is a positive integer. By the proof of Theorem 3,

$$\begin{aligned} \mathcal{Z}_{\mathcal{A}} &= \left\{ S^{-i}\beta_j \mid 1 \leq i \leq \left[\frac{\beta_j}{\lambda} \right] \right\}_{j=1}^{s} \\ &= \left\{ S^{-(i-1)}\alpha_j \mid 0 \leq i - 1 \leq m_i \right\}_{j=1}^{s} \\ &= \mathcal{C}. \end{aligned}$$

□

Example 2. In Example 1 we found the reduced λ_5-BQFs in a G_5-equivalence class $\mathcal{A}$. The forms

$$Q_2 = [3\lambda + 4, -11\lambda - 3, \lambda + 2],$$

and

$$Q_5 = [\lambda + 2, -13\lambda - 5, 11\lambda + 8],$$

are the only reduced forms $Q = [A, B, C]$ in $\mathcal{A}$ which correspond to reduced numbers greater than λ. The simple forms in $\mathcal{A}$ are all related to Q_2 and Q_5 by S^j as

$$\begin{aligned}
Q_2 \circ S &= [3\lambda + 4, 3\lambda + 3, -3\lambda - 2], \\
Q_5 \circ S &= [\lambda + 2, -7\lambda - 3, -3\lambda - 2], \\
Q_5 \circ S^2 &= [\lambda + 2, -\lambda - 1, -9\lambda - 6], \text{and} \\
Q_5 \circ S^3 &= [\lambda + 2, 5\lambda + 1, -7\lambda - 4].
\end{aligned}$$

We could calculate the corresponding simple numbers directly from these λ_5-BQFs. Instead, we will use the reduced numbers we found in Example 1 along with the proof of Theorem 3. The simple numbers in $\mathcal{A}$ are

$$\begin{aligned}
\mathcal{Z}_{\mathcal{A}} &= \left\{ S^{-1}\beta_2, S^{-1}\beta_5, S^{-2}\beta_5, S^{-3}\beta_5 \right\} \\
&= \left\{ [1; \overline{1, 1, 4, 2}], [3; \overline{2, 1, 1, 4}], [2; \overline{2, 1, 1, 4}], [1; \overline{2, 1, 1, 4}] \right\} \\
&= \left\{ \frac{-3\lambda - 3 + \sqrt{D}}{6\lambda + 8}, \frac{7\lambda + 3 + \sqrt{D}}{2\lambda + 4}, \frac{\lambda + 1 + \sqrt{D}}{2\lambda + 4}, \frac{-5\lambda - 1 + \sqrt{D}}{2\lambda + 4} \right\} \\
&= \left\{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \right\},
\end{aligned}$$

where $D = 135\lambda + 86 = \frac{307 + 135\sqrt{5}}{2}$. Finally, these simple numbers form a finite orbit of Φ_5. We have

$$\begin{aligned}
\Phi_5(\alpha_1) &= TU\alpha_1 = \alpha_2, \\
\Phi_5(\alpha_2) &= TU^4\alpha_2 = \alpha_2 - \lambda = \alpha_3, \\
\Phi_5(\alpha_3) &= TU^4\alpha_3 = \alpha_3 - \lambda = \alpha_4, \\
\Phi_5(\alpha_4) &= TU^3\alpha_4 = \alpha_1.
\end{aligned}$$

Acknowledgments

I thank Barbara Nimershiem for sharing her geometric insight in many stimulating discussions. I thank the referee for reading carefully and for making valuable suggestions about an earlier manuscript.

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