

# A Hecke Correspondence Theorem for Automorphic Integrals with Symmetric Rational Period Functions on the Hecke Groups

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## Abstract

Marvin Knopp developed the theory of automorphic integrals, which generalize automorphic forms; each automorphic integral has an additional period function in its automorphic relation. The period functions satisfy relations that arise from the underlying group relations. Knopp showed that entire automorphic integrals with rational period functions satisfy a Hecke correspondence theorem, provided the rational period functions have poles only at 0 or  $\infty$ . For other automorphic integrals each corresponding Dirichlet series has a functional equation with a remainder term that arises from the nonzero poles of the rational period function.

In this paper we prove a Hecke correspondence theorem for a class of automorphic integrals with rational period functions on the Hecke groups. We restrict our attention to automorphic integrals of weight that is twice an odd integer and to rational period functions that satisfy a symmetry property we call “Hecke-symmetry.” We explicate the relationship between the structure of the rational period functions and the corresponding remainder terms. Each remainder term satisfies two relations (the second of which is new in this paper) corresponding to the two relations for the rational period function.

## 1 Introduction

In the 1930s Hecke [8, 9] formalized a general correspondence between automorphic forms and Dirichlet series. Hecke’s work generalized Riemann’s use of the transformation law for the elliptic  $\theta$ -function to derive the functional equation for the zeta function  $\zeta(s)$  in [20].

Hecke studied automorphic forms with respect to an infinite class of discrete groups that act on the upper half plane as linear fractional transformations. These groups have become known as the Hecke groups, and include the modular group  $\Gamma(1) = \text{PSL}(2, \mathbb{Z})$ .

In [5] Eichler introduced generalized abelian integrals, which he obtained by integrating modular forms of positive weight. An Eichler integral satisfies a modular relation with a polynomial period function. In [11] and [12] Knopp generalizes Eichler integrals and develops the theory of automorphic integrals with rational period functions. Knopp shows that an entire modular integral corresponds to a Dirichlet series that satisfies Hecke's functional equation, provided the rational period function has poles only at 0 or  $\infty$ . Knopp also proves a converse theorem, from which it follows that if a rational period function has any other poles the corresponding Dirichlet series cannot satisfy the same functional equation.

In [7] Hawkins and Knopp prove a Hecke correspondence theorem for modular integrals with rational period functions on  $\Gamma_\theta$ , the theta subgroup of  $\Gamma(1)$ . In this correspondence the Dirichlet series functional equation contains a remainder term that corresponds to the nonzero poles of the rational period function. Hawkins and Knopp observe that their theorem implies that an automorphic integral with a rational period function on one of the Hecke groups must correspond to a Dirichlet series that satisfies a functional equation similar to the one they found. Hawkins and Knopp also point out that since the Hecke groups have two group relations, while  $\Gamma_\theta$  has a single group relation, rational period functions on Hecke groups have more structure than rational period functions on  $\Gamma_\theta$ . Thus for Hecke groups the corresponding remainder terms must have more structure than the ones discovered by Hawkins and Knopp, a fact that a full correspondence theorem in this setting must reveal.

In [3] this author proves a Hecke correspondence theorem for modular integrals with rational period functions on  $\Gamma(1)$ , which is one of the Hecke groups. We show that the remainder term for the Dirichlet series functional equation satisfies a second relation that corresponds to the second group relation in  $\Gamma(1)$ .

In this paper we extend the correspondence to a class of automorphic integrals with rational period functions on *any* of the Hecke groups. We restrict our attention to automorphic integrals of weight that is twice an odd integer and to rational period functions that satisfy a certain symmetry property we call "Hecke-symmetry." We show that the remainder term in the Dirichlet series functional equation satisfies a second relation that generalizes the second relation in [3].

## 2 Hecke groups and fixed points

Let  $\lambda$  be a fixed positive real number and put  $S = S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ ,  $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , and  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Define the group  $G(\lambda) = \langle S, T \rangle / \{\pm I\} \subseteq \text{PSL}(2, \mathbb{R})$ . Elements of this group act on the Riemann sphere as linear fractional transformations, that is,  $Mz = \frac{az+b}{cz+d}$  for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$  and  $z \in \mathbb{C} \cup \{\infty\}$ . This action preserves the real line and the upper half-plane  $\mathcal{H}$ .

Erich Hecke [8, 9] showed that the values of  $\lambda$  between 0 and 2 for which  $G(\lambda)$  is discrete are

$$\lambda = \lambda_p = 2 \cos(\pi/p),$$

for  $p = 3, 4, 5, \dots$ . These discrete groups are the *Hecke groups*, which we denote by  $G_p = G(\lambda_p)$  for  $p \geq 3$ . Each of the Hecke groups has two group relations, which may be written  $T^2 = (ST)^p = I$ . (Note that we are identifying  $I$  and  $-I$ , since these are projective groups.) The first of these groups is the modular group  $G_3 = G(1) = \Gamma(1)$ . For the rest of the paper we fix the integer  $p \geq 3$  and the real number  $\lambda = \lambda_p$ .

An element  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$  is *hyperbolic* if  $|a+d| > 2$ , *parabolic* if  $|a+d| = 2$ , and *elliptic* if  $|a+d| < 2$ . We designate fixed points accordingly. The element  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  fixes

$$\begin{aligned} z &= \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c} \\ &= \frac{a - d \pm \sqrt{(a+d)^2 - 4}}{2c}, \end{aligned} \tag{1}$$

so hyperbolic elements of  $G_p$  have two distinct real fixed points. Since  $G_p$  is discrete, the stabilizer of any complex number is a cyclic subgroup of  $G_p$  [15, page 15]. If  $\alpha$  is a hyperbolic fixed point of  $G_p$  we call the other point fixed by its stabilizer the *Hecke conjugate* of  $\alpha$ , and we denote it by  $\alpha'$ . A straightforward calculation shows that if  $\alpha$  is hyperbolic and  $M \in G_p$ , then  $(M\alpha)' = M\alpha'$ . If  $R$  is a set of hyperbolic fixed points of  $G_p$  we write  $R' = \{x' \mid x \in R\}$ . We say that  $R$  has *Hecke symmetry* if  $R = R'$ .

### 3 Automorphic integrals

Suppose  $F$  is a function holomorphic in the upper half-plane  $\mathcal{H}$  with the Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z / \lambda}, \tag{2}$$

for  $z \in \mathcal{H}$ . If for every  $z \in \mathcal{H}$ ,  $F$  satisfies the automorphic relation

$$z^{-2k} F\left(\frac{-1}{z}\right) = F(z) + q(z), \tag{3}$$

where  $q(z)$  is a rational function and  $2k \in 2\mathbb{Z}^+$ , we say that  $F$  is an *entire automorphic integral of weight  $2k$  on  $G_p$  with rational period function (RPF)  $q$* . If  $q \equiv 0$  then  $F$  is an entire automorphic form of weight  $2k$  on  $G_p$ . For  $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G_p$  and  $F(z)$  a complex function, we define the *weight  $2k$  slash operator*  $F|_{2k} M = F|_M$  by

$$(F|_M)(z) = (cz + d)^{-2k} F(Mz).$$

With this notation we may rewrite the automorphic relation (3) as

$$F|_T = F + q.$$

A calculation shows that  $F \mid M_1 M_2 = (F \mid M_1) \mid M_2$  for  $M_1, M_2 \in G_p$ . We use this and the group relation  $T^2 = I$  to calculate that a RPF  $q$  satisfies the relation

$$q \mid T + q = 0. \quad (4)$$

In a similar way the relation  $(ST)^p = I$  implies that  $q$  satisfies a second relation

$$q + q \mid ST + \cdots + q \mid (ST)^{p-1} = 0. \quad (5)$$

Knopp [10, Section II] showed that (4) and (5) characterize the set of RPFs for a given group and weight.

## 4 Binary quadratic forms

Hawkins [6] pointed out a deep connection between RPFs on the modular group and classical binary quadratic forms. Schmidt [22] and Schmidt and Sheingorn [23] observed that similar connections exist between RPFs on the Hecke groups and binary quadratic forms with coefficients in  $\mathbb{Z}[\lambda_p]$ . We give properties of these binary quadratic forms in [19] and we exploit the connections to RPFs on the Hecke groups in [4]. In this section we describe the results we need to describe the structure of RPFs and to develop our correspondence.

We consider binary quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2,$$

with coefficients in  $\mathbb{Z}[\lambda]$ . We denote such a form by  $Q = [A, B, C]$  and refer to it as a  $\lambda_p$ -BQF or a  $\lambda$ -BQF. We restrict our attention to indefinite forms, which have positive discriminant  $D = B^2 - 4AC$ .

Elements of a Hecke group act on  $\lambda$ -BQFs by  $(Q \circ M)(x, y) = Q(ax+by, cx+dy)$  for  $Q$  a  $\lambda$ -BQF and  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$ . Since  $Q \circ M = Q \circ (-M)$ , this definition does not depend on our choice of coset representative. A calculation shows that  $B'^2 - 4A'C' = B^2 - 4AC$ , so the action of a Hecke group preserves the discriminant. We say that  $Q$  and  $\tilde{Q}$  are  $G_p$ -equivalent, and write  $Q \sim \tilde{Q}$ , if there exists a  $V \in G_p$  such that  $\tilde{Q} = Q \circ V$ .  $G_p$ -equivalence is an equivalence relation, so  $G_p$  partitions the set of  $\lambda$ -BQFs into equivalence classes of forms.

In [19] we describe a one-to-one correspondence between hyperbolic fixed points of  $G_p$  and certain  $\lambda_p$ -BQFs. In order to do so we use a variant of Rosen's  $\lambda$ -continued fractions [21, 22] to first map every hyperbolic point  $\alpha$  to the unique primitive hyperbolic element  $M_\alpha \in G_p$  with positive trace that has  $\alpha$  as an attracting fixed point. This mapping associates Hecke conjugates with inverse elements in the Hecke group, that is,  $M_{\alpha'} = M_\alpha^{-1}$ . We then map every primitive hyperbolic element  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_p$  with positive trace to the  $\lambda$ -BQF  $Q = [c, d-a, -b]$ . The roots of  $Q(z, 1) = cz^2 + (d-a)z - b$  are the fixed points of  $M$  given by (1). Since every domain element for this map is hyperbolic we call the images *hyperbolic*  $\lambda$ -BQFs. We also note that every hyperbolic  $\lambda$ -BQF is indefinite. The composition of the two maps above associates every hyperbolic fixed point  $\alpha$  with a unique hyperbolic  $\lambda$ -BQF that we denote  $Q_\alpha$ . These

mappings are all injective, so the inverse of the composition exists; this inverse associates every hyperbolic quadratic form  $Q = [A, B, C]$  with the hyperbolic number  $\alpha_Q = \frac{-B+\sqrt{D}}{2A}$ , where  $D$  is the discriminant of  $Q$ .

Suppose that  $V \in G_p$  and  $\alpha$  and  $\beta$  are hyperbolic numbers. Then  $\beta = V^{-1}\alpha$  if and only if  $Q_\beta = Q_\alpha \circ V$  for the associated forms, and if and only if  $M_\beta = V^{-1}M_\alpha V$  for the associated matrices. Now trace is preserved by conjugation, as is the property of a matrix being primitive. Thus every  $G_p$ -equivalence class of  $\lambda$ -BQFs contains either only hyperbolic forms or no hyperbolic forms, so we designate  $\lambda$ -BQF equivalence classes themselves as hyperbolic or not hyperbolic.

If a hyperbolic quadratic form  $Q = [A, B, C]$  satisfies  $A > 0 > C$  we say that  $Q$  is  $G_p$ -simple (or *simple*, if the context is clear). If  $Q$  is  $G_p$ -simple, we say that the associated hyperbolic fixed point  $\alpha_Q$  is a  $G_p$ -simple (or *simple*) number. A hyperbolic number  $\alpha$  is simple if and only if  $\alpha > 0 > \alpha'$ . If  $\mathcal{A}$  is a hyperbolic equivalence class of  $\lambda$ -BQFs we write  $\mathcal{Z}_\mathcal{A} = \{\alpha_Q \mid Q \in \mathcal{A}, G_p\text{-simple}\}$ , the set of associated simple numbers.

Suppose that  $\alpha$  is a hyperbolic number and that  $\mathcal{A}$  is a hyperbolic equivalence class of  $\lambda$ -BQFs. Then  $-\mathcal{A} = \{[-A, -B, -C] \mid [A, B, C] \in \mathcal{A}\}$  is also an equivalence class of  $\lambda$ -BQFs, not necessarily distinct from  $\mathcal{A}$ . A calculation shows that  $Q_{\alpha'} = -Q_\alpha$ , so  $Q_\alpha \in \mathcal{A}$  if and only if  $Q_{\alpha'} \in -\mathcal{A}$ .

## 5 Rational period functions

Choi and Zagier [1] and Parson [17] gave an explicit characterization of RPFs on the modular group that made possible the correspondence in [3]. The problem of finding a characterization of RPFs on all of the Hecke groups remains open, although several authors have done work in this direction [4, 16, 19, 22, 23]. These results provide enough information about the structure of RPFs on the Hecke groups for us to prove our correspondence theorem.

In this section we summarize the results about RPFs on the Hecke groups. We give properties of RPF poles, quote a characterization of RPFs for the case we are considering, and describe modifications that emphasize the second relation (5).

Throughout this section we suppose that  $q$  is an RPF of weight  $2k \in 2\mathbb{Z}^+$  on  $G_p$  with pole set  $\mathcal{P} = \mathcal{P}(q)$ . We will not assume that  $k$  is odd or that  $q$  has a Hecke-symmetric set of poles until subsection 5.2; the results of subsection 5.1 hold for all RPFs on  $G_p$ .

### 5.1 Poles

Hawkins [6] introduced the idea of an *irreducible system of poles* (or *irreducible pole set*), the minimal set of RPF poles, which must occur together because of the relations (4) and (5). Meier and Rosenberger [16] observed that all the poles of  $q$  are real and Schmidt [22] proved that all nonzero poles are hyperbolic fixed points of  $G_p$ .

Let  $\mathcal{P}^+$  and  $\mathcal{P}^-$  denote the positive and negative poles, respectively, in  $\mathcal{P}$ , and put  $\mathcal{P}^* = \mathcal{P}^+ \cup \mathcal{P}^-$ . Schmidt [22] showed that  $\mathcal{P}^*$  is a disjoint union of “cycle pairs.” We show in [4] that each of Schmidt’s cycle pairs can be written as  $\mathcal{Z}_{\mathcal{A}} \cup T\mathcal{Z}_{\mathcal{A}}$ , where  $\mathcal{A}$  is a hyperbolic equivalence class of  $\lambda$ -BQFs and  $T\mathcal{Z}_{\mathcal{A}} = \{T\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}}\}$ . Each  $\mathcal{Z}_{\mathcal{A}} \cup T\mathcal{Z}_{\mathcal{A}}$  is an irreducible system of poles; the poles in  $\mathcal{Z}_{\mathcal{A}}$  are positive and the poles in  $T\mathcal{Z}_{\mathcal{A}}$  are negative. In [4] we also rewrite the negative poles in an irreducible system of poles as  $T\mathcal{Z}_{\mathcal{A}} = \mathcal{Z}'_{-\mathcal{A}}$ , where  $\mathcal{Z}'_{-\mathcal{A}} = \{\alpha' \mid \alpha \in \mathcal{Z}_{\mathcal{A}}\}$ . As a result we may write any irreducible system of poles as  $\mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{-\mathcal{A}}$  for some hyperbolic  $\lambda$ -BQF equivalence class  $\mathcal{A}$ . A set  $\mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{-\mathcal{A}}$  has Hecke symmetry if and only if  $\mathcal{A} = -\mathcal{A}$ , so a Hecke symmetric irreducible system of poles has the form  $\mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{\mathcal{A}}$ . If all of the irreducible systems of poles of a RPF are Hecke-symmetric we say that the RPF itself is Hecke-symmetric.

## 5.2 RPF characterization

Meier and Rosenberger [16] showed that if an RPF of weight  $2k \in 2\mathbb{Z}^+$  on  $G_p$  has a pole only at zero, it must be of the form

$$q_0(z) = \begin{cases} \nu(1 - z^{-2k}), & \text{if } 2k \neq 2, \\ \nu(1 - z^{-2}) + \eta z^{-1}, & \text{if } 2k = 2, \end{cases} \quad (6)$$

for constants  $\nu$  and  $\eta$ .

Let  $k$  be an odd positive integer and write  $Q_{\alpha}(z) = Q_{\alpha}(z, 1)$  for each  $\lambda$ -BQF  $Q$ . Suppose that  $q$  is a Hecke-symmetric RPF of weight  $2k$  on  $G_p$ . In [4] we show that every such RPF is of the form

$$q(z) = \sum_{\ell=1}^L d_{\ell} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_{\ell}}} Q_{\alpha}(z)^{-k} + c_0 q_0(z),$$

where each  $\mathcal{A}_{\ell}$  is a hyperbolic  $G_p$ -equivalence class of  $\lambda$ -BQFs, the  $d_{\ell}$  ( $1 \leq \ell \leq L$ ) are constants, and  $q_0(z)$  is given by (6). We will see that the part of  $q$  with nonzero poles determines the remainder term, so we write

$$q(z) = q^*(z) + c_0 q_0(z), \quad (7)$$

where

$$q^*(z) = \sum_{\ell=1}^L d_{\ell} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_{\ell}}} Q_{\alpha}(z)^{-k} \quad (8)$$

has poles

$$P^* = \bigcup_{\ell=1}^L (\mathcal{Z}_{\mathcal{A}_{\ell}} \cup \mathcal{Z}'_{\mathcal{A}_{\ell}}).$$

Since both  $q$  and  $c_0 q_0$  are RPFs,  $q^*$  is an RPF as well and satisfies the relations (4) and (5). We show in [4, Lemma 9] that

$$\frac{D^{k/2}}{Q_{\alpha}(z)^k} = \frac{(\alpha - \alpha')^k}{(z - \alpha)^k (z - \alpha')^k}, \quad (9)$$

where  $D$  is the discriminant of the  $\lambda$ -BQF  $Q_\alpha$ . Thus we may write  $q^*$  and  $q$  more explicitly as

$$q^*(z) = \sum_{\ell=1}^L d_\ell D_\ell^{-k/2} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} \frac{(\alpha - \alpha')^k}{(z - \alpha)^k (z - \alpha')^k}, \quad (10)$$

and

$$q(z) = \sum_{\ell=1}^L d_\ell D_\ell^{-k/2} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} \frac{(\alpha - \alpha')^k}{(z - \alpha)^k (z - \alpha')^k} + c_0 q_0(z). \quad (11)$$

## 6 The direct theorem

In this section we show that an entire automorphic integral of positive even weight on one of the Hecke groups may be associated with a Dirichlet series satisfying a functional equation. We restrict our attention to automorphic integrals of weight  $2k$  ( $k$  odd) with Hecke-symmetric RPFs. We show that the Dirichlet series functional equation involves a remainder term, which comes from the RPF for the automorphic integral.

Let  $k$  be an odd positive integer. Suppose that  $F$  is an entire automorphic integral of weight  $2k \in 2\mathbb{Z}^+$  on  $G_p$  with Hecke-symmetric RPF  $q$ . We may assume without loss of generality that  $F$  is a *cusp* automorphic integral, that is,  $a_0 = 0$  in the Fourier expansion (2).

Write  $z = x + iy$  with  $x, y \in \mathbb{R}$ . It can be shown [10, 622-623] that  $F$  satisfies

$$|F(z)| \leq K (|z|^\alpha + y^{-\beta}), \quad z \in \mathcal{H} \quad (12)$$

for some positive real numbers  $K$ ,  $\alpha$  and  $\beta$ . It follows that the coefficients  $a_n$  in the Fourier expansion (2) for  $F$  satisfy

$$a_n = \mathcal{O}(n^\beta), \quad n \rightarrow +\infty. \quad (13)$$

This, with  $a_0 = 0$  in (2), implies that

$$F(iy) = \mathcal{O}(e^{-2\pi y/\lambda}), \quad y \rightarrow +\infty. \quad (14)$$

The growth estimates (12) and (14) allow us to define the *Mellin transform* of  $F$ ,

$$\Phi(s) = \int_0^\infty F(iy) y^s \frac{dy}{y}, \quad (15)$$

a function of the complex variable  $s = \sigma + it$ . This integral converges for  $\sigma > \beta$ . For  $\sigma > \beta + 1$ , we can integrate term by term to get

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \phi(s), \quad (16)$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (17)$$

is the *Dirichlet series associated with F*. The bound on the growth of the Fourier coefficients  $a_n$  (13) implies that the sum in (17) converges absolutely and uniformly on compact subsets of the right half plane  $\sigma > \beta + 1$ , so that  $\phi(s)$  is analytic there.

We invert part of the Mellin transform of  $F$  and use the automorphic relation (3) and the RPF decomposition (7) to get

$$\begin{aligned} \int_0^1 F(iy)y^s \frac{dy}{y} &= \int_1^{\infty} F\left(\frac{-1}{iy}\right) y^{-s} \frac{dy}{y} \\ &= - \int_1^{\infty} F(iy)y^{2k-s} \frac{dy}{y} - \int_1^{\infty} c_0 q_0(iy)y^{2k-s} \frac{dy}{y} \\ &\quad - \int_1^{\infty} q^*(iy)y^{2k-s} \frac{dy}{y}. \end{aligned}$$

Thus

$$\Phi(s) = D(s) + E^0(s) + E^*(s),$$

where

$$D(s) = \int_1^{\infty} F(iy) [y^s - y^{2k-s}] \frac{dy}{y}, \quad (18)$$

$$E^0(s) = - \int_1^{\infty} c_0 q_0(iy)y^{2k-s} \frac{dy}{y}, \quad (19)$$

and

$$E^*(s) = - \int_1^{\infty} q^*(iy)y^{2k-s} \frac{dy}{y}. \quad (20)$$

Now  $D(s)$  is entire and satisfies the functional equation

$$D(2k - s) + D(s) = 0. \quad (21)$$

The expression (6) for  $q_0$  implies that  $q_0(z) = \mathcal{O}(1)$  as  $|z| \rightarrow \infty$ , so the integral defining  $E^0(s)$  in (19) converges in the right half-plane  $\sigma > 2k$ . The expression (8) for  $q^*$  implies that  $q^*(z) = \mathcal{O}(|z|^{-2k})$  as  $|z| \rightarrow \infty$ , so the integral defining  $E^*(s)$  in (20) converges in the right half-plane  $\sigma > 0$ .

In order to write the functional equation for  $\Phi(s)$  that is suggested by (21) we need meromorphic continuations of  $E^0(s)$  and  $E^*(s)$  to the  $s$ -plane. We use (6) to calculate that

$$E^0(s) = \begin{cases} -\tilde{a}_0 \left( \frac{1}{s-2k} + \frac{1}{s} \right), & \text{if } 2k \neq 2, \\ -\tilde{a}_0 \left( \frac{1}{s-2} + \frac{1}{s} \right) + \frac{\tilde{b}_1 i}{s-1}, & \text{if } 2k = 2, \end{cases} \quad (22)$$

where  $\tilde{a}_0 = a_0 c_0$  and  $\tilde{b}_0 = b_0 c_0$ . In every case  $E^0(s)$  has a meromorphic continuation to all of the complex  $s$ -plane, with simple poles at  $s = 0, 2k$  (and at

$s = 1$  if  $2k = 2$ ). Furthermore,  $E^0(s)$  satisfies the same functional equation as  $D(s)$ ,

$$E^0(2k - s) + E^0(s) = 0. \quad (23)$$

For the meromorphic continuation of  $E^*(s)$  we first use a partial fraction decomposition of the right side of (9)

$$\frac{(\alpha - \alpha')^k}{(z - \alpha)^k (z - \alpha')^k} = \sum_{m=1}^k \frac{a_{m,\alpha}}{(z - \alpha)^m} + \sum_{n=1}^k \frac{b_{n,\alpha'}}{(z - \alpha')^n},$$

where  $a_{m,\alpha} = (-1)^{m-k} \binom{2k-m-1}{k-1} (\alpha - \alpha')^{m-k}$  and  $b_{n,\alpha'} = (-1)^k \binom{2k-n-1}{k-1} (\alpha - \alpha')^{n-k}$ . This along with (8) gives us

$$q^*(z) = \sum_{\ell=1}^L c_\ell \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} \left( \sum_{m=1}^k \frac{a_{m,\alpha}}{(z - \alpha)^m} + \sum_{n=1}^k \frac{b_{n,\alpha'}}{(z - \alpha')^n} \right), \quad (24)$$

where  $c_\ell = d_\ell D_\ell^{-k/2}$  with  $D_\ell$  the discriminant of the BQFs in the equivalence class  $\mathcal{A}_\ell$ .

We use the integral representation for the hypergeometric function [14, equation (9.1.6)]

$${}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 y^{b-1} (1-y)^{c-b-1} (1-yz)^{-a} dy, \quad (25)$$

for  $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$  and  $|\arg(1-z)| < \pi$ . We let  $a = m$ ,  $b = m - s$ , and  $c = 1 + m - s$  for  $s$  a complex variable and  $m$  a positive integer, and we use a change of variables to invert, so

$$\int_1^\infty \frac{y^s}{(y-z)^m} \frac{dy}{y} = \frac{1}{r-s} {}_2F_1[m, m-s; 1+m-s; z],$$

for  $\sigma < m$  and  $|\arg(1-z)| < \pi$ . We let  $z = i\alpha$  for  $\alpha \in \mathbb{R}$  and multiply by  $i^{-m}$ , and get

$$\int_1^\infty \frac{y^s}{(iy - \alpha)^m} \frac{dy}{y} = \frac{i^{-m}}{m-s} {}_2F_1[m, m-s; 1+m-s; -i\alpha], \quad (26)$$

for  $\sigma < m$ .

The expression (24) and formula (26) together imply that  $E^*(s)$  is a finite linear combination of terms of the form

$$\int_1^\infty \frac{y^{2k-s}}{(iy - \alpha)^m} \frac{dy}{y} = \frac{i^{-m}}{s-2k+m} {}_2F_1[m, s-2k+m; s-2k+m+1; -i\alpha], \quad (27)$$

for  $\alpha \in \mathcal{P}^*$ ,  $1 \leq m \leq k$ , and  $\sigma > 2k - m$ . Now the hypergeometric function  ${}_2F_1[a, b; c; z]$  is an entire function of  $a$  and  $b$ , and a meromorphic function of  $c$  with simple poles at  $c = 0, -1, -2, \dots$ . Thus the function in (27) is meromorphic

in the  $s$ -plane with simple poles at  $s = 2k - m, 2k - m - 1, 2k - m - 2, \dots$ . Since  $E^*(s)$  is a finite linear combination of terms of the form (27) with  $1 \leq m \leq k$ , we have that  $E^*(s)$  is a meromorphic function with simple poles at most at  $s = 2k - 1, 2k - 2, \dots$ . Thus  $\Phi(s)$  is meromorphic in the  $s$ -plane.

We now show that  $\Phi(s)$  is bounded in lacunary vertical strips of the form

$$S(\sigma_1, \sigma_2; t_0) = \{s = \sigma + it \mid \sigma_1 \leq \sigma \leq \sigma_2, |t| \geq t_0 > 0\}. \quad (28)$$

Now  $D(s)$  is bounded in every vertical strip since the integral in (18) converges for every  $s$ ,  $|y^s| = y^\sigma \geq y^{\sigma_2}$  and  $|y^{2k-s}| = y^{2k-\sigma} \leq y^{2k-\sigma_1}$ , and the integrand is unchanged as  $t \rightarrow \infty$ . Also, the expression (22) shows that  $E^0(s)$  is bounded in every lacunary vertical strip, since its poles are excluded from every  $S(\sigma_1, \sigma_2; t_0)$  and each term approaches zero as  $t \rightarrow \infty$ .

It remains to show that  $E^*(s)$  is bounded in every lacunary vertical strip. Since  $E^*(s)$  is a finite sum of terms of the form (27), it is sufficient to prove that for every purely imaginary  $\beta$ , and for  $1 \leq m \leq k$  the function

$$\frac{1}{s - 2k + m} {}_2F_1[m, s - 2k + m; s - 2k + m + 1; \beta]$$

is bounded in any  $S(\sigma_1, \sigma_2; t_0)$ . Given any lacunary vertical strip of the form (28) we use (25) to write

$$\begin{aligned} \frac{1}{s - 2k + m} {}_2F_1[m, s - 2k + m; s - 2k + m + 1; \beta] \\ = \int_0^1 y^{s-2k+m-1} (1 - \beta y)^{-m} dy, \end{aligned}$$

for  $\sigma > 2k - m$  and  $\alpha \in \mathbb{R}$ . This integral is bounded in every  $S(\sigma_1, \sigma_2; t_0)$  for which  $\sigma_1 > 2k - m$ , since  $|y^{s-2k+m-1} (1 - \beta y)^{-m}| \leq y^{\sigma_1-2k+m-1}$  for  $s \geq \sigma_1$  and  $0 \leq y \leq 1$ . If  $\sigma_1 \leq 2k - m$  we let  $n \in \mathbb{Z}^+$  such that  $\sigma_1 > 2k - m - n$  and we integrate by parts  $n$  times. The result is

$$\begin{aligned} \frac{1}{s - 2k + m} {}_2F_1[m, s - 2k + m; s - 2k + m + 1; \beta] \\ = \sum_{j=1}^n \frac{(-1)^{j+1} \Gamma(s - 2k + m) m! \beta^{j-1}}{\Gamma(s - 2k + m) (1 - \beta)^{m+j-1} (m - j + 1)!} \\ + \frac{(-1)^{n+1} \Gamma(s - 2k + m) m! \beta^n}{\Gamma(s - 2k + m + n) (m - n)!} \int_0^1 y^{s-2k+m+n-1} (1 - \beta y)^{-m-n} dy, \end{aligned} \quad (29)$$

for  $\sigma > 2k - m - n$ . This integral is bounded in  $S(\sigma_1, \sigma_2; t_0)$  since  $|y^{s-2k+m+n-1} (1 - \beta y)^{-m-n}| \leq y^{\sigma_1-2k+m+n-1}$  for  $s \geq \sigma_1$  and  $0 \leq y \leq 1$ . The other expressions on the right-hand side of (29) are rational in  $s$  with simple poles at integer values, and they each approach zero as  $t \rightarrow \infty$ . Thus the function in (29) is bounded in every  $S(\sigma_1, \sigma_2; t_0)$ , which implies that  $E^*(s)$  and  $\Phi(s)$  are bounded there as well.

Since  $\Phi(s)$  is meromorphic we may write the functional equation suggested by (21),

$$\Phi(2k - s) + \Phi(s) = R(s), \quad (30)$$

where  $R(s)$  is a meromorphic function we call the *remainder term*. By (21) and (23) we have

$$R(s) = E^*(2k - s) + E^*(s), \quad (31)$$

so the remainder term depends only on  $q^*(z)$ , the part of the RPF with nonzero poles. The expression (30) (or (31)) implies that  $R(s)$  satisfies the (first) relation

$$R(2k - s) - R(s) = 0. \quad (32)$$

We must calculate an explicit expression for  $R(s)$ , in order to give meaning to (30) and to prove the converse theorem. If we use the fact that  $q^*$  satisfies the first relation (4) to replace  $q^*(iy)$  in (20) and invert, we have

$$E^*(s) = - \int_0^1 q^*(iy) y^s \frac{dy}{y}. \quad (33)$$

On the other hand, if we substitute directly into (20) we have

$$E^*(2k - s) = - \int_1^\infty q^*(iy) y^s \frac{dy}{y}. \quad (34)$$

The integral in (33) converges for  $\sigma > 0$  since  $q^*(iy)$  is bounded as  $y \rightarrow 0$ , and the integral in (34) converges for  $\sigma < 2k$  since  $q^*(iy) = \mathcal{O}(y^{-2k})$  as  $y \rightarrow \infty$ . Thus for  $0 < \sigma < 2k$  we have

$$R(s) = - \int_0^\infty q^*(iy) y^s \frac{dy}{y}, \quad (35)$$

that is,  $R(s)$  is the negative of the Mellin transform of  $q^*(z)$ . This expression makes it clear that the first relation for  $q^*(z)$  leads directly to the first relation for  $R(s)$ . If we use (4) to replace  $q^*(iy)$  in (35) and invert the variable of integration we get (32).

We will substitute the expression for  $q^*$  given by (10) into (35) and write  $R(s)$  as a linear combination of integrals of the form

$$\int_0^\infty \frac{y^s}{(iy - \alpha)^k (iy - \alpha')^k} \frac{dy}{y},$$

which converge for  $0 < \sigma < 2k$ . The evaluation of these integrals involves exponential functions of the form  $z^a = e^{a \log z}$ , where  $\log z = \log |z| + i \arg z$  for  $z \in \mathbf{C}$ . We will take the principal branch for each logarithm, using the convention that  $-\pi \leq \arg z < \pi$ . We need the integral formula in the following lemma, which uses the beta function  $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$  and the hypergeometric function.

**Lemma 1.** Let  $k \in \mathbb{Z}^+$ , and let  $\delta$  and  $\epsilon$  be nonzero real numbers that satisfy one of:  $\delta < 0 < \epsilon$ ,  $\epsilon < 0 < \delta$ ,  $0 < \epsilon < \delta$ , or  $\delta < \epsilon < 0$ . Then

$$\begin{aligned} & \int_0^\infty \frac{y^s}{(iy - \delta)^k (iy - \epsilon)^k} \frac{dy}{y} \\ &= i^s \delta^{s-k} B(2k - s, s - k) (\delta - \epsilon)^{-k} {}_2F_1 \left[ k, 1 - k; k - s + 1; \frac{\epsilon}{\epsilon - \delta} \right] \\ & \quad + i^s \epsilon^{s-k} B(s, k - s) (\delta - \epsilon)^{-k} {}_2F_1 \left[ k, 1 - k; s - k + 1; \frac{\epsilon}{\epsilon - \delta} \right], \end{aligned} \quad (36)$$

for  $0 < \operatorname{Re} s < 2k$ .

*Proof.* At several places the proof below involves branching questions that reduce to calculations of the form  $(z_1 z_2)^s = z_1^s z_2^s$ , which is valid if  $\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$  using our argument convention.

We start with (25) and change variables by letting  $y = \frac{u}{u+1}$ . We also let  $z = 1 - v/w$ , where  $v$  is a positive real number and  $w \in \mathbf{C} \setminus \mathbf{R}$ , which implies that  $|\arg(1 - z)| = |\arg(v/w)| < \pi$ . The result is

$$\int_0^\infty u^{b-1} (u+1)^{a-c} (w+vu)^{-a} du = w^{-a} B(b, c-b) {}_2F_1[a, b; c; 1 - v/w],$$

for  $\operatorname{Re} c > \operatorname{Re} b > 0$ ,  $v > 0$  and  $w \in \mathbf{C} \setminus \mathbf{R}$ . We change variables again by letting  $u = y/v$ , and put  $a = k$ ,  $b = s$ , and  $c = 2k$  ( $k \in \mathbf{Z}^+$  and  $s \in \mathbf{C}$ ) so that

$$\int_0^\infty \frac{y^s}{(y+v)^k (y+w)^k} \frac{dy}{y} = v^{s-k} w^{-k} B(s, 2k-s) {}_2F_1[k, s; 2k; 1 - v/w]$$

for  $0 < \sigma = \operatorname{Re}(s) < 2k$ ,  $v > 0$  and  $w \in \mathbf{C} \setminus \mathbf{R}$ . We will apply the two identities for hypergeometric functions [14, equations (9.5.3) and (9.5.9)],

$${}_2F_1[a, b; c; z] = (1-z)^{c-a-b} {}_2F_1[c-a, c-b; c; z], \quad (37)$$

for  $|\arg(1-z)| < \pi$ , and

$$\begin{aligned} {}_2F_1[a, b; c; z] &= (-z)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} {}_2F_1[a, 1+a-c; 1+a-b; 1/z] \\ & \quad + (-z)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} {}_2F_1[b, 1+b-c; 1+b-a; 1/z], \end{aligned} \quad (38)$$

for  $|\arg(-z)| < \pi$  and  $|\arg(1-z)| < \pi$ . We apply (38) with  $a = k$ ,  $b = s$ ,  $c = 2k$  and  $z = 1 - v/w$ , and get

$$\begin{aligned} & \int_0^\infty \frac{y^s}{(y+v)^k (y+w)^k} \frac{dy}{y} \\ &= v^{s-k} (v-w)^{-k} B(2k-s, s-k) {}_2F_1 \left[ k, 1-k; k-s+1; \frac{w}{w-v} \right] \\ & \quad + v^{s-k} w^{s-k} (v-w)^{-s} B(s, k-s) {}_2F_1 \left[ s, 1+s-2k; s-k+1; \frac{w}{w-v} \right], \end{aligned}$$

for  $0 < \sigma < 2k$ ,  $v > 0$  and  $w \in \mathbf{C} \setminus \mathbf{R}$ . Next we apply (37) to the hypergeometric function in the second term, with  $a = s$ ,  $b = 1 + s - 2k$ ,  $c = s - k + 1$  and  $z = \frac{w}{w-v}$ . After simplifying we have

$$\begin{aligned} & \int_0^\infty \frac{y^s}{(y+v)^k(y+w)^k} \frac{dy}{y} \\ &= v^{s-k}(v-w)^{-k} B(2k-s, s-k) {}_2F_1 \left[ k, 1-k; k-s+1; \frac{w}{w-v} \right] \\ & \quad + w^{s-k}(v-w)^{-k} B(s, k-s) {}_2F_1 \left[ k, 1-k; s-k+1; \frac{w}{w-v} \right], \quad (39) \end{aligned}$$

for  $0 < \sigma < 2k$ ,  $v > 0$  and  $w \in \mathbf{C} \setminus \mathbf{R}$ . Next we restrict  $w$  to  $\arg w = \pm\pi/2$  and put  $v = \delta i$  and  $w = \epsilon i$  in (39). A simplification gives us (36) for  $\arg \delta = -\pi/2$  and  $\epsilon \in \mathbf{R}$ ,  $\epsilon \neq 0$ .

To complete the proof we must consider the possible values of  $\delta$  and  $\epsilon$ . We consider  $\delta$  to be a complex variable and do an analytic continuation of (36) in  $\delta$  to the region  $-\pi \leq \arg \delta < \pi/2$  subject to the restriction that  $\left| \arg \left( 1 - \left( \frac{\epsilon}{\delta} \right) \right) \right| = \left| \arg \left( \frac{\delta}{\delta-\epsilon} \right) \right| < \pi$ . If we let  $\delta \in \mathbf{R}$ , this restriction is  $\frac{\delta}{\delta-\epsilon} > 0$ , which implies the restrictions given in the statement of the Lemma.  $\square$

We calculate  $R(s)$  explicitly by substituting (10) into (35) to get

$$R(s) = - \sum_{\ell=1}^L d_\ell D_\ell^{-k/2} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} (\alpha - \alpha')^k \int_0^\infty \frac{y^s}{(iy - \alpha)^k (iy - \alpha')^k} \frac{dy}{y},$$

for  $0 < \sigma < 2k$  with  $D_\ell$  the discriminant of the  $\lambda$ -BQFs in  $\mathcal{A}_\ell$ . We may apply Lemma 1 to the integrals in this expression, since  $\alpha > 0 > \alpha'$ . This gives us

$$\begin{aligned} R(s) &= - \sum_{\ell=1}^L d_\ell D_\ell^{-k/2} i^s \\ & \quad \times \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} \left\{ \alpha^{s-k} B(2k-s, s-k) {}_2F_1 \left[ k, 1-k; k-s+1; \frac{\alpha'}{\alpha - \alpha'} \right] \right. \\ & \quad \left. + (\alpha')^{s-k} B(s, k-s) {}_2F_1 \left[ k, 1-k; s-k+1; \frac{\alpha'}{\alpha' - \alpha} \right] \right\}. \quad (40) \end{aligned}$$

We may use this expression to verify directly that the remainder term satisfies the first relation (32). We use (40) to write  $R(2k-s)$ , then use the fact that  $\alpha \in \mathcal{Z}_{\mathcal{A}}$  implies  $-\frac{1}{\alpha} = \gamma'$  and  $-\frac{1}{\alpha'} = \gamma \in \mathcal{Z}_{\mathcal{A}}$ , as well as a calculation that  $\frac{\alpha'}{\alpha' - \alpha} = \frac{\gamma'}{\gamma' - \gamma}$  to obtain  $R(s)$ .

We have proved the following theorem.

**Theorem 1.** *Fix  $p \geq 3$ , let  $\lambda = \lambda_p$ , and let  $k$  be an odd positive integer. Suppose that  $F(z)$  is an entire automorphic integral of weight  $2k$  on  $G_p$  with*

Hecke-symmetric rational period function  $q(z)$  given by (8). Suppose that  $F$  has the Fourier expansion (2) with zero constant term, and that  $\Phi(s)$  is given by (15) for  $\text{Re}(s) > \beta$ , for some positive  $\beta$ .

Then for  $\text{Re}(s) > \beta + 1$   $\Phi(s)$  is also given by (16) and (17), and  
(a)  $\Phi(s)$  has a meromorphic continuation to the  $s$ -plane with, at worst, simple poles at integer points  $m \leq 2k$ ; for  $\text{Re}(s) > \beta$  we have

$$\Phi(s) = D(s) + E^0(s) + E^*(s);$$

$D(s)$  is given by (18) and is entire, while  $E^0(s)$  and  $E^*(s)$  are given by (19) and (20) (respectively) and have meromorphic continuations to the  $s$ -plane; furthermore,

- (b)  $\Phi(s)$  is bounded in every lacunary vertical strip of the form (28), and  
(c)  $\Phi(s)$  satisfies the functional equation (30) with  $R(s)$  given by (40).

## 7 The second relation

We have observed that the remainder term  $R(s)$  satisfies one relation (32). In this section we show that  $R(s)$  must satisfy a second relation, which follows from the fact that the corresponding RPF  $q^*(z)$  satisfies the second relation (5). We first modify (8) in order to exhibit the fact that  $q^*$  satisfies (5).

We let  $U = ST = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$  and focus on the effect of  $U$  on the poles of an RPF. Schmidt proves in [22] that if  $\alpha \in \mathcal{P}^*$ , then  $U^m \alpha \in \mathcal{P}^*$  for exactly one  $m$ ,  $1 \leq m \leq p-1$ . Thus we may separate the poles in  $\mathcal{P}^*$  into pairs that are images of each other under some power of  $U$ . For each pair of poles we will determine the power  $m$  from the sizes of the poles.

A calculation shows that

$$0 = U^p(0) < U^{p-1}(0) < \dots < U^2(0) < U(0) = \infty,$$

so we may partition the extended real line into a disjoint union of  $p$  half-open intervals,

$$[-\infty, \infty) = [-\infty, U^p(0)) \cup [U^p(0), U^{p-1}(0)) \cup \dots \cup [U^2(0), U(0)). \quad (41)$$

$U$  maps each interval to the previous interval, the first to the last, and left endpoints to left endpoints. We denote the  $j$ th interval of (41) by  $I_j$ , that is  $I_j = [U^{p-j+2}(0), U^{p-j+1}(0))$  for  $1 \leq j \leq p$ . We note that all negative poles are in  $I_1$  and each positive pole is in one of the  $I_j$ ,  $2 \leq j \leq p$ .

**Lemma 2.** Fix  $p \geq 3$  and  $2 \leq j \leq p$ . For every  $G_p$ -equivalence class  $\mathcal{A}$  of  $\lambda_p$ -BQFs we have  $\{\beta' \mid \beta \in \mathcal{Z}_{-\mathcal{A}} \cap I_{p-j+2}\} = \{U^{j-1}\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}} \cap I_j\}$ , where  $I_j$  is the  $j$ th interval of (41) and  $\beta'$  is the Hecke conjugate of  $\beta$ .

*Proof.* We show containment in both directions. First suppose that  $\gamma \in \{\beta' \mid \beta \in \mathcal{Z}_{-\mathcal{A}} \cap I_{p-j+2}\}$ , so  $\gamma = \beta'$  for some  $\beta$  in both  $\mathcal{Z}_{-\mathcal{A}}$  and  $I_{p-j+2}$ . Let  $\alpha = U^{p-j+1}\gamma$ , so  $\gamma = U^{j-1}\alpha$ . We need to show that  $\alpha$  is in both  $\mathcal{Z}_{\mathcal{A}}$  and  $I_j$ . Now  $\beta$  is simple, so  $\gamma = \beta' \in I_1$ ,

which implies that  $\alpha = U^{-(j-1)}\gamma \in I_j$ . We also have  $\gamma' = \beta \in I_{p-j+2}$ , so  $\alpha' = U^{p-j+1}\gamma' \in I_1$ . Thus  $\alpha' < 0 < \alpha$ , so  $\alpha$  is simple. Now  $\beta \in \mathcal{Z}_{-\mathcal{A}}$  implies that  $Q_\beta \in -\mathcal{A}$ , so  $Q_\gamma = Q_{\beta'} \in \mathcal{A}$ . But since  $\alpha = U^{p-j+1}\gamma$  this means that  $Q_\alpha \in \mathcal{A}$ . Thus  $\alpha \in \mathcal{Z}_{\mathcal{A}}$  and we have shown that  $\{\beta' \mid \beta \in \mathcal{Z}_{-\mathcal{A}} \cap I_{p-j+2}\} \subseteq \{U^{j-1}\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}} \cap I_j\}$ .

The demonstration of containment in the other direction is a similar calculation.  $\square$

Lemma 2 means that every negative pole of an RPF is connected by a power of  $U$  to a positive pole. Specifically, if  $\beta' \in \mathcal{P}^-$ , then  $\beta' = U^{j-1}\alpha$  for some  $\alpha \in \mathcal{P}^+$ . The power  $j-1$  is determined by the size of  $\beta$ , the Hecke conjugate of  $\beta'$ . Although Lemma 2 holds for all equivalence classes of  $\lambda_p$ -BQFs, we will apply it to equivalence classes that have Hecke-symmetry. In this setting equivalence classes satisfy  $-\mathcal{A} = \mathcal{A}$ , so we have  $\{\beta' \mid \beta \in \mathcal{Z}_{\mathcal{A}} \cap I_{p-j+2}\} = \{U^{j-1}\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}} \cap I_j\}$ .

Since  $k$  is odd and  $Q_\alpha = -Q_{\alpha'}$  we have

$$\sum_{\alpha \in \mathcal{Z}_{\mathcal{A}}} Q_\alpha(z)^{-k} = - \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}}} Q_{\alpha'}(z)^{-k}.$$

Using this to replace half of  $q^*$  in (8) we have

$$q^*(z) = \sum_{\ell=1}^L \frac{d_\ell}{2} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}} (Q_\alpha(z)^{-k} - Q_{\alpha'}(z)^{-k}).$$

Now every  $\alpha \in \mathcal{Z}_{\mathcal{A}_\ell}$  is positive and thus in one of the intervals  $I_j$ ,  $2 \leq j \leq p$ , of (41). We write  $q^*$  in a way that exhibits these intervals as

$$q^*(z) = \sum_{\ell=1}^L \frac{d_\ell}{2} \sum_{j=2}^p \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \cap I_j} (Q_\alpha(z)^{-k} - Q_{\alpha'}(z)^{-k}). \quad (42)$$

Now  $2 \leq j \leq p$  if and only if  $2 \leq p-j+2 \leq p$ , so

$$\begin{aligned} \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \cap I_j} Q_{\alpha'}(z)^{-k} &= \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \cap I_{p-j+2}} Q_{\alpha'}(z)^{-k} \\ &= \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \cap I_j} Q_{U^{j-1}\alpha}(z)^{-k}. \end{aligned}$$

We have used Lemma 2 for the second equality. We relabel the constant for each irreducible system of poles and combine this with (42) to write

$$q^*(z) = \sum_{\ell=1}^L c_\ell \sum_{j=2}^p \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_\ell} \cap I_j} (Q_\alpha(z)^{-k} - Q_{U^{j-1}\alpha}(z)^{-k}). \quad (43)$$

We turn our attention to the second relation for the remainder term. We define

$$R(s; a, b) = (a-b)^k \int_0^\infty \frac{y^s}{(iy-a)^k (iy-b)^k} \frac{dy}{y}, \quad (44)$$

for  $s$  a complex variable with  $0 < \operatorname{Re}(s) < 2k$  and for  $a$  and  $b$  nonzero real numbers. We substitute (43) into (35) and use (9) to write an alternative expression for the remainder term,

$$R(s) = - \sum_{\ell=1}^L c_{\ell} D_{\ell}^{-k/2} \sum_{j=2}^p \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_{\ell}} \cap I_j} (R(s; \alpha, \alpha') - R(s; U^{j-1}\alpha, U^{j-1}\alpha')). \quad (45)$$

We define a mapping acting on expressions of the form (44) by

$$\rho(R(s; a, b)) = -R(2k - s; a - \lambda, b - \lambda). \quad (46)$$

We extend the action of  $\rho$  to linear combinations of terms of the form (44) by linearity, and note that  $\rho$  is of order  $p$ . The next lemma will allow us to write the second relation for  $R(s)$  using  $\rho$ .

**Lemma 3.**  $-R(2k - s; a - \lambda, b - \lambda) = R(s; U^{-1}a, U^{-1}b)$

The proof of the lemma uses the integral definition (44), a change of variables, and some simple manipulations.

**Theorem 2.** Fix  $p \geq 3$ , let  $\lambda = \lambda_p$ , and let  $\rho$  be the mapping defined by (46). Suppose that  $R(s)$  is a remainder term for the Dirichlet series that corresponds to an entire modular integral with a Hecke-symmetric rational period function on  $G_p$ . Then  $R(s)$  satisfies the (second) relation

$$R + \rho(R) + \rho^2(R) + \cdots + \rho^{p-1}(R) = 0. \quad (47)$$

*Proof.* Suppose that  $\hat{R}$  is an expression in the domain of  $\rho$ . Then a calculation shows that for any positive integer  $m$  the expression  $\hat{R} - \rho^m(\hat{R})$  satisfies the second relation (47). We use Lemma 3 to rewrite the expression (45) for  $R(s)$  as

$$R(s) = - \sum_{\ell=1}^L c_{\ell} D_{\ell}^{-k/2} \sum_{j=2}^p \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}_{\ell}} \cap I_j} (R(s; \alpha, \alpha') - \rho^{p-j+1}(R(s; \alpha, \alpha'))).$$

Thus  $R(s)$  is a linear combination of terms of the form

$$\hat{R} - \rho^m(\hat{R}),$$

so  $R(s)$  satisfies (47). □

## 8 The converse theorem

We now prove the following converse to Theorem 1.

**Theorem 3.** Fix  $p \geq 3$ , let  $\lambda = \lambda_p$ , and let  $k$  be an odd positive integer. Suppose the Dirichlet series (17) converges absolutely in the half-plane  $\text{Re}(s) > \gamma$ . Suppose that the function  $\Phi(s)$  defined by (16) satisfies

(a)  $\Phi(s)$  has a meromorphic continuation to the  $s$ -plane with, at worst, simple poles at integer points;

(b)  $\Phi(s)$  is bounded in every lacunary vertical strip of the form (28) and

(c)  $\Phi(s)$  satisfies the functional equation (30) with  $R(s)$  given by (40). Then  $\phi(s)$  is the Dirichlet series associated with an entire automorphic integral of weight  $2k$  on  $G_p$  with Hecke-symmetric rational period function given by (11) and (6).

*Proof.* We write  $s = \sigma + it$ , with  $\sigma, t \in \mathbb{R}$ . Since (17) converges absolutely in the half-plane  $\sigma > \gamma$ , we have  $a_n = \mathcal{O}(n^{\gamma-1})$ , so  $F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z / \lambda}$  converges for  $z \in \mathcal{H}$ . We follow Riemann [20] and Hecke [8, 9], take the inverse Mellin transform of  $\Phi(s)$  and rearrange to get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi(s) y^{-s} ds = \sum_{n=1}^{\infty} a_n e^{-2\pi n y / \lambda} = F(iy),$$

for any  $c > 0$ . The interchange of the sum and integral is valid by Stirling's formula [2, p. 224]

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma-1/2} e^{-\pi|t|/2}, \quad (48)$$

as  $|t| \rightarrow \infty$ . Let  $M$  be a positive integer with  $M > \gamma$  and  $M > 2k$ , and fix  $c$  with  $M < c < M + 1$ . We move the line of integration from  $\sigma = c$  to  $\sigma = 2k - c$ . In order to do so we use the fact that

$$\lim_{|T| \rightarrow \infty} \int_{2k-c+iT}^{c+iT} \Phi(s) y^{-s} ds = 0,$$

which follows from the fact that  $\Phi(s)$  is bounded in the lacunary vertical strip  $S(2k - c, c; t_0)$ , along with Stirling's formula (48) and the Phragmén Lindelöf Theorem [24, p. 180]. The integrand  $\Phi(s)y^{-s}$  has possible poles at the integers between the lines of integration, except it does not have a pole at  $s = M$  since  $\phi(s)$  converges absolutely for  $\sigma = M$ . We pick up the residues at these poles, so

$$F(iy) = \frac{1}{2\pi i} \int_{2k-c-i\infty}^{2k-c+i\infty} \Phi(s) y^{-s} ds + \frac{1}{2\pi i} \sum_{m=2k-M}^{M-1} \text{Res}_{s=m} \{ \Phi(s) y^{-s} \}.$$

We use the functional equation (30) and a change of variables to calculate that

$$(F | T)(iy) - F(iy) = -\frac{1}{2\pi i} \int_{2k-c-i\infty}^{2k-c+i\infty} R(s) y^{-s} ds - \frac{1}{2\pi i} \sum_{m=2k-M}^{M-1} \text{Res}_{s=m} \{ \Phi(s) \} y^{-m}. \quad (49)$$

In order to evaluate this integral we invert the formula in Lemma 1. Since the integral in (36) converges absolutely for  $0 < \sigma < 2k$  and since  $\frac{1}{(iy-\delta)^k(iy-\epsilon)^k}$  is of bounded variation, we have the inverse Mellin transform [25, Theorem 9a]

$$\begin{aligned} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} \{i^s \delta^{s-k} B(2k-s, s-k) {}_2F_1 \left[ k, 1-k; k-s+1; \frac{\epsilon}{\epsilon-\delta} \right] \\ + i^s \epsilon^{s-k} B(s, k-s) {}_2F_1 \left[ k, 1-k; s-k+1; \frac{\epsilon}{\epsilon-\delta} \right]\} y^{-s} ds \\ = \frac{(\delta-\epsilon)^k}{(iy-\delta)^k(iy-\epsilon)^k}, \quad (50) \end{aligned}$$

for  $0 < d < 2k$  and  $y > 0$ , and for  $\delta < 0 < \epsilon$ ,  $\epsilon < 0 < \delta$ ,  $0 < \epsilon < \delta$ , or  $\delta < \epsilon < 0$ . We move the line of integration for the integral in (49) from  $\sigma = 2k - c$  to  $\sigma = 1/2$ , and pick up the negatives of the residues of  $R(s)y^{-s}$  at  $s = 2k - M, 2k - M + 1, \dots, 0$ . We may do so because

$$\lim_{|T| \rightarrow \infty} \int_{2k-c+iT}^{1/2+iT} R(s)y^{-s} ds = 0,$$

which follows from the fact the the Beta functions in (40) have exponential decay as  $|t| \rightarrow \infty$  and all of the other functions are bounded as  $|t| \rightarrow \infty$ . Thus

$$\begin{aligned} (F | T)(iy) - F(iy) = -\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} R(s)y^{-s} ds + \frac{1}{2\pi i} \sum_{m=2k-M}^0 \text{Res}_{s=m} \{R(s)\} y^{-m} \\ - \frac{1}{2\pi i} \sum_{m=2k-M}^{M-1} \text{Res}_{s=m} \{\Phi(s)\} y^{-m}. \quad (51) \end{aligned}$$

We substitute (40) into the integral and apply (50) to get

$$\begin{aligned} (F | T)(iy) - F(iy) &= \sum_{\ell=1}^L d_\ell D_\ell^{-k/2} \sum_{\alpha \in \mathcal{Z}_{A_\ell}} \frac{(\alpha - \alpha')^k}{(iy - \alpha)^k (iy - \alpha')^k} \\ &+ \frac{1}{2\pi i} \sum_{m=2k-M}^0 \text{Res}_{s=m} \{R(s)\} y^{-m} \\ &- \frac{1}{2\pi i} \sum_{m=2k-M}^{M-1} \text{Res}_{s=m} \{\Phi(s)\} y^{-m} \\ &= q^*(iy) + \sum_{m=2k-M}^0 a_m (iy)^{-m} + \sum_{m=2k-M}^{M-1} b_m (iy)^{-m}, \end{aligned}$$

where  $a_m = \frac{i^m}{2\pi i} \text{Res}_{s=m} \{R(s)\}$  and  $b_m = \frac{-i^m}{2\pi i} \text{Res}_{s=m} \{\Phi(s)\}$ . The identity theorem gives us

$$(F | T)(z) - F(z) = q(z),$$

for  $z \in \mathcal{H}$ , where

$$q(z) = q^*(z) + \sum_{m=2k-M}^0 a_m z^{-m} + \sum_{m=2k-M}^{M-1} b_m z^{-m},$$

is a rational function of  $z$ . Thus  $F$  is an entire automorphic integral of weight  $2k$  on  $G_p$  with RPF  $q$ . Now  $q^*$  is a Hecke-symmetric RPF, so  $q(z) - q^*(z)$  must be an RPF with a pole only at zero. Thus  $q(z) - q^*(z)$  must have the form (6).  $\square$

*Remarks.* (i) The fact that  $q(z) - q^*(z)$  has the form given in (6) means that  $b_m = 0$  for  $2k < m < M$ . This implies that  $\Phi(s)$  cannot have a pole at any of the values  $s = 2k + 1, 2k + 2, \dots, M - 1$ . This is an additional restriction since we originally assumed only that  $\phi(s)$  converges absolutely in some half-plane  $\text{Re}(s) > \gamma$ .

(ii) The fact that  $q(z) - q^*(z)$  has the form given in (6) also means that  $a_m + b_m = 0$  for  $2k - M \leq m < 2k$  with  $m \neq 0$  (and  $m \neq 1$  if  $2k = 2$ ). This implies that for these values of  $m$

$$\text{Res}_{s=m} \{R(s) - \Phi(s)\} = 0. \quad (52)$$

The fact that this holds for  $2k - M \leq m < 0$  implies that even though  $\Phi(s)$  and  $R(s)$  both have poles at negative integer values, the residues must all cancel.

(iii) If we apply the functional equation (30) to (52) we get

$$\text{Res}_{s=m} \{\Phi(2k - s)\} = 0,$$

for  $2k - M \leq m < 2k$  with  $m \neq 0$  (and  $m \neq 1$  if  $2k = 2$ ), or

$$\text{Res}_{s=n} \{\Phi(n)\} = 0,$$

for  $0 < n \leq M$  with  $n \neq 2k$  (and  $n \neq 1$  if  $2k = 2$ ). For  $2k < n < M$  this is our observation in (i).

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