A HECKE CORRESPONDENCE THEOREM FOR MODULAR INTEGRALS WITH RATIONAL PERIOD FUNCTIONS *

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1 Introduction

In the 1930's Erich Hecke used the Mellin transform and its inverse to demonstrate a systematic relationship between automorphic forms and Dirichlet series [5, 6]. In particular, entire modular forms on the full modular group $\Gamma(1) = SL(2, \mathbf{Z})$ correspond to Dirichlet series which satisfy a functional equation.

In [2] Eichler introduced generalized abelian integrals which he obtained by integrating modular forms of positive weight. An Eichler integral satisfies a modular relation with a polynomial period function. In [8, 9] Marvin Knopp generalized Eichler integrals and developed the theory of modular integrals with rational period functions.

In [9] Knopp shows that an entire modular integral with a rational period function corresponds to a Dirichlet series which satisfies Hecke's functional equation, provided the rational period function has poles only at 0 or ∞. Knopp also proves a converse theorem, from which it follows that if the rational period function has any other poles the corresponding Dirichlet series does not satisfy the same functional equation.

In [4] Hawkins and Knopp prove a Hecke correspondence theorem in which a modular integral with an arbitrary rational period function corresponds to a Dirichlet series which satisfies a more general functional equation. In this case the functional equation for the Dirichlet series contains an additional remainder term which arises from the poles of the rational period function which are not

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at 0 or ∞. Hawkins and Knopp formulate their results for modular integrals on the theta group, Γ_θ, a subgroup of index 3 in $\Gamma(1)$. The theta group has a single group relation and any rational period function on Γ_θ must satisfy a corresponding relation. This relation in turn imposes a relation on the remainder term in the functional equation for the corresponding Dirichlet series.

In this paper we present a Hecke correspondence theorem for modular integrals of weight $2k \in 2\mathbf{Z}^+$ with rational period functions on the *full* modular group $\Gamma(1)$. The modular group has a second group relation which imposes more structure (than Γ_θ) on any modular integral, forcing its rational period function to satisfy a second relation. This in turn imposes more structure on the remainder term in the functional equation for the corresponding Dirichlet series. We will show that a remainder term associated with $\Gamma(1)$ must satisfy a second relation which arises from the second relation for the rational period function and we will write the remainder term and its second relation explicitly.

A modular integral on $\Gamma(1)$ is also a modular integral on Γ_θ, so we will use the results of Hawkins and Knopp in this setting. In particular, a modular integral on $\Gamma(1)$ corresponds to a Dirichlet series which satisfies a functional equation with a remainder term arising from the poles of the rational period function which are not at 0 or ∞.

In Section 2 we define modular integrals and rational period functions. In Section 3 we describe the characterization of rational period functions on $\Gamma(1)$ given by Choie and Zagier [1] (and, independently, by Parson [11]), which we modify in order to emphasize the second relation. In Section 4 we prove the direct Hecke theorem, in which a modular integral with an arbitrary rational period function on $\Gamma(1)$ gives rise to a Dirichlet series with an associated remainder term. Our theorem includes an explicit characterization of such remainder terms. In Section 5 we describe the second relation which a remainder term must satisfy. We complete the correspondence in Section 6, where we prove the converse theorem.

2 Modular integrals

We will consider $\Gamma(1) = SL(2,\mathbf{Z})$ to be a group of linear fractional transformations acting on $\mathcal{H}$, the upper half plane, by putting $Mz = \frac{az+b}{cz+d}$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and $z \in \mathcal{H}$. With this interpretation we identify an element M with its negative $-M$. $\Gamma(1)$ is generated by $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and satisfies the group relations

$$T^2 = (ST)^3 = I.$$

Suppose F is a function holomorphic in $\mathcal{H}$ and has the Fourier expansion

$$F(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \; z \in \mathcal{H}. \tag{1}$$

Let $2k \in 2\mathbf{Z}$. If for every $z \in \mathcal{H}$ and $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1)$, F satisfies the modular relation

$$(cz+d)^{-2k} F(Mz) = F(z) + q_M(z), \tag{2}$$

we say that F is an *entire modular integral of weight* $2k$ *on* $\Gamma(1)$. We call each q_M the *period function of* F *corresponding to* $M \in \Gamma(1)$. In this paper we will only consider period functions which are rational. If for every $M \in \Gamma(1)$ we have that $q_M \equiv 0$, then F is an entire modular form of weight $2k$ on $\Gamma(1)$.

Using the *slash operator* $F \mid_{2k} M = F \mid M$ defined by

$$(F \mid M)(z) = (cz+d)^{-2k} F(Mz),$$

we may rewrite (2) as

$$F \mid M = F + q_M. \tag{3}$$

A calculation shows that for any function F which is defined on $\mathcal{H}$ and M_1, $M_2 \in \Gamma(1)$,

$$F \mid M_1 M_2 = (F \mid M_1) \mid M_2,$$

from which it follows that

$$q_{M_1 M_2} = q_{M_1} \mid M_2 + q_{M_2}.$$

Since $\Gamma(1)$ is generated by S and T, the rational period functions for a modular integral F are generated by q_S and q_T. However, the Fourier expansion (1) implies that $F \mid S = F$, so that $q_S = 0$. Thus (3) is equivalent to $F \mid T = F + q_T$. We call $q = q_T$ *the* rational period function for F, and say that F satisfies the modular relation

$$F \mid T = F + q. \tag{4}$$

The group relation $T^2 = I$ implies that a rational period function q satisfies the relation

$$q \mid T + q = 0, \tag{5}$$

and the relation $(ST)^3 = I$ implies that q satisfies the second relation

$$q \mid (ST)^2 + q \mid ST + q = 0. \tag{6}$$

Knopp [7, ∮ II] showed that any rational function satisfying (5) and (6) is in fact the rational period function for a modular integral. In other words, (5) and (6) characterize the set of rational period functions for a given weight.

3 Rational period functions

This section summarizes the results we need concerning the characterization of rational period functions on $\Gamma(1)$. It also describes modifications, in which we write each rational period function in a way that emphasizes the second relation (6). We do this in order to describe the second relation for the remainder term in the functional equation of the corresponding Dirichlet series.

In [9] Marvin Knopp proves that the poles of a rational period function on $\Gamma(1)$ occur only at 0, ∞, or at real quadratic irrationalities. He also shows that when the weight $2k$ is positive and the period function has only rational poles, it is of the form

$$q(z) = \begin{cases} c_1(1-z^{-2}) + c_2 z^{-1} & ,k=1 \\ c(1-z^{-2k}) & ,k>1 \end{cases}, \tag{7}$$

where c, c_1, and c_2 are complex numbers. The function $c(1-z^{-2k})$ (for *any* $k \in \mathbf{R}$) is the period function for the trivial modular integral $F(z) \equiv -c$. The function $c_2 z^{-1}$ is a multiple of the period function for $E_2(z)$, the Eisenstein series of weight 2 on $\Gamma(1)$.

In [3] Hawkins describes the pole set of a rational period function and shows that it is the disjoint union of irreducible systems of poles. If $q(z)$ has a pole at a fixed quadratic irrational number α, an *irreducible system of poles*, $P(\alpha)$, is the minimal set of quadratic irrational numbers which must be poles of $q(z)$ because of (5) and (6). Hawkins also observes a connection between irreducible pole sets and indefinite binary quadratic forms.

We will use the following definitions and properties of quadratic forms which can be found in [13]. Let A, B and C be relatively prime integers such that $D = B^2 - 4AC$ is positive and not a square. Then $Q(x,y) = Ax^2 + Bxy + Cy^2$ is called a *primitive indefinite binary quadratic form* of *discriminant* D. We also denote $Q(x,y)$ by $Q = [A, B, C]$.

Given a quadratic form $Q(x,y)$ of discriminant D and a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, define

$$\hat{Q}(x,y) = (Q \circ M)(x,y) = Q(ax+by, cx+dy),$$

which is another quadratic form of the same discriminant D. Suppose that Q and $\hat{Q}$ are two binary quadratic forms of discriminant D. We will say that Q and $\hat{Q}$ are *equivalent in the narrow sense*, and write $Q \sim \hat{Q}$, if there is an element M of $\Gamma(1)$ such that $\hat{Q} = Q \circ M$. The relation $Q \sim \hat{Q}$ is an equivalence relation (narrow equivalence) on the set of quadratic forms of a given discriminant. Let $\mathcal{A}$ denote a narrow equivalence class of binary quadratic forms and define another (not necessarily distinct) equivalence class of forms,

$$\theta\mathcal{A}^{-1} = \{[-A, -B, -C] : [A, B, C] \in \mathcal{A}\}.$$

We will use a natural correspondence between binary quadratic forms and real quadratic irrational numbers. The form $Q = [A, B, C]$ is associated with $\alpha = \alpha_Q = \frac{B+\sqrt{D}}{2A}$, one of the roots of $Q(z, -1) = Az^2 - Bz + C$. We will write $Q \leftrightarrow \alpha$ to denote this correspondence. The other root $\alpha' = \frac{B-\sqrt{D}}{2A}$ is associated with the negative form $-Q = [-A, -B, -C]$.

A quadratic form is said to be *simple* if $A > 0 > C$. Quadratic irrational numbers associated with simple forms are also said to be *simple*. A quadratic irrationality α is simple if and only if $\alpha > 0 > \alpha'$, where α' is the algebraic conjugate of α. A quadratic form is said to be *reduced* if $A > 0$, $C > 0$, and $B > A + C$. *Reduced* quadratic irrational numbers are those associated with reduced forms. A quadratic irrationality α is reduced if and only if $\alpha > 1 > \alpha' > 0$. Each narrow equivalence class $\mathcal{A}$ contains a finite, positive number of reduced forms [13].

Hawkins proves in [3] that an irreducible pole set $P(\alpha)$ corresponds to the set of reduced quadratic forms in a narrow equivalence class. Since any class $\mathcal{A}$ of forms corresponds to a unique irreducible pole set we will also denote the pole set by $P(\mathcal{A})$.

Choie and Zagier [1] establish a connection between the simple quadratic forms in an equivalence class $\mathcal{A}$ and the pole set $P(\mathcal{A})$. Let $\mathcal{Z}_\mathcal{A}$ denote the set of simple quadratic irrational numbers which are associated with $\mathcal{A}$.

Lemma 1 (Choie and Zagier) $P(\mathcal{A}) = \mathcal{Z}_\mathcal{A} \cup T\mathcal{Z}_\mathcal{A}$.

It is worth noting that $\mathcal{Z}_\mathcal{A}$ is the set of positive poles in $P(\mathcal{A})$ and that $T\mathcal{Z}_\mathcal{A} = \{-1/\alpha \mid \alpha \in \mathcal{Z}_\mathcal{A}\}$ is the set of negative poles in $P(\mathcal{A})$.

Choie and Zagier show that any rational period function q of weight $2k$ has the form

$$q(z) = \sum_{\mathcal{A}} C_\mathcal{A} \sum_{\alpha \in \mathcal{Z}_\mathcal{A}} (q_\alpha(z) - q_{T\alpha}(z)) + q_0'(z). \tag{8}$$

The outer sum is on the (finite number of) classes of binary quadratic forms which correspond to the irreducible pole sets of q. Each $C_\mathcal{A}$ is a complex number which depends only on the class $\mathcal{A}$. The functions q_α and $q_{T\alpha}$ are the principal parts of q at α and $T\alpha$, respectively, normalized so the coefficients of $(z-\alpha)^{-k}$ in q_α and $(z-T\alpha)^{-k}$ in $q_{T\alpha}$ are both one. The function q_0' is a rational function with a pole only at zero of order at most $2k$.

A complete description of the rational period function $q(z)$ requires explicit expressions for the functions q_α and $q_{T\alpha}$. Let $PP_\alpha[f]$ denote the principle part of $f(z)$ at $z = \alpha$. Choie and Zagier prove the following lemma.

Lemma 2 (Choie and Zagier) *Let α be a quadratic irrationality, α' its conjugate. Then*

$$q_\alpha(z) = PP_\alpha \left[\frac{(\alpha - \alpha')^k}{(z-\alpha)^k (z-\alpha')^k} \right] = PP_\alpha \left[\frac{D^{k/2}}{(az^2 - bz + c)^k} \right], \tag{9}$$

where $[a,b,c]$ *is the binary quadratic form associated to* α *and* D *is the discriminant of* $[a,b,c]$.

The proof of Lemma 2 also shows that under the same assumptions,

$$q_{\alpha'}(z) = PP_{\alpha'}\left[\frac{(\alpha'-\alpha)^k}{(z-\alpha)^k(z-\alpha')^k}\right] = PP_{\alpha'}\left[\frac{(-1)^k D^{k/2}}{(az^2-bz+c)^k}\right], \tag{10}$$

an expression which we will use later. We may write q_α and $q_{\alpha'}$ in a more explicit way, using the partial fraction decomposition

$$\begin{aligned}\frac{1}{(az^2-bz+c)^k} &= \frac{1}{a^k}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha'-\alpha)^{l-2k}(-1)^k}{(z-\alpha)^l} \\ &\quad + \frac{1}{a^k}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha-\alpha')^{l-2k}(-1)^k}{(z-\alpha')^l}.\end{aligned}$$

We have

$$q_\alpha(z) = \sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha-\alpha')^{l-k}(-1)^{l-k}}{(z-\alpha)^l}, \tag{11}$$

and

$$q_{\alpha'}(z) = \sum_{l=1}^{k}\binom{2k-1-l}{k-l}\frac{(\alpha-\alpha')^{l-k}}{(z-\alpha')^l}. \tag{12}$$

We will modify the characterization of rational period functions given by Choie and Zagier in order to emphasize the second relation. We begin with an alternative way to express an irreducible pole set $P(\mathcal{A})$, in which we write the negative poles as algebraic conjugates of the positive poles. Let $\mathcal{Z}'_{\mathcal{A}}$ denote $\{\alpha' : \alpha \in \mathcal{Z}_{\mathcal{A}}\}$.

Lemma 3 $P(\mathcal{A}) = \mathcal{Z}_{\mathcal{A}} \cup \mathcal{Z}'_{\theta\mathcal{A}^{-1}}$.

Proof: A routine argument demonstrating containment in both directions shows that $T\mathcal{Z}_{\mathcal{A}} = \mathcal{Z}'_{\theta\mathcal{A}^{-1}}$. This, along with Lemma 1, completes the proof. $\square$

We will rewrite the part of a rational period function which corresponds to the poles in $\mathcal{Z}_{\mathcal{A}}$ which are between zero and one. The following lemma will allow us to distinguish these poles when using the assosiated quadratic forms.

Lemma 4 *Suppose that* α *is a simple quadratic irrational number associated with the quadratic form* $[a,b,c]$*. Then*

(i) $\alpha > 1$ *if and only if* $b > a+c$*, and*

(ii) $0 < \alpha < 1$ *if and only if* $b < a+c$*.*

The proof of Lemma 4 is a routine exercise in using inequalities.

The next lemma will allow us to rewrite those poles in $P(\mathcal{A})$ which are images under $(ST)^2$ of other poles in $P(\mathcal{A})$. Since $(ST)^{-1} = (ST)^2$ these are the poles whose images under ST are in $P(\mathcal{A})$.

Lemma 5 *For every equivalence class $\mathcal{A}$ of quadratic forms we have*

(i) $\{\beta \mid \beta \in \mathcal{Z}_{\mathcal{A}}, 0 < \beta < 1\} = \left\{(ST)^2\alpha' \mid \alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}}, \alpha > 1\right\}$, *and*

(ii) $\{\beta' \mid \beta \in \mathcal{Z}_{\theta\mathcal{A}^{-1}}, 0 < \beta < 1\} = \left\{(ST)^2\alpha \mid \alpha \in \mathcal{Z}_{\mathcal{A}}, \alpha > 1\right\}$.

Proof: Statement (i) is equivalent to statement (ii), as we can see if we replace every element with its algebraic conjugate and interchange the class names $\mathcal{A}$ and $\theta\mathcal{A}^{-1}$. An argument which involves containment in both directions and some tedious manipulations shows that statement (i) is true. $\square$

Choie and Zagier observe, in a somewhat different form [1, page 95], that for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$,

$$q_{M\alpha} = q_\alpha \mid M^{-1} - PP_{a/c}\left[q_\alpha \mid M^{-1}\right], \tag{13}$$

where $PP_{a/c}\left[q_\alpha \mid M^{-1}\right](z)$ has a pole at $z = a/c$ of order at most $2k-1$. If $M = (ST)^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, then $M^{-1} = ST = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $a/c = 0$. Thus we have

$$q_{(ST)^2\alpha} = q_\alpha \mid ST - PP_0\left[q_\alpha|ST\right], \tag{14}$$

where $PP_0\left[q_\alpha|ST\right]$ has a pole at $z = 0$ of order at most $2k-1$.

We may now express any rational period function on $\Gamma(1)$ in a way that emphasizes the second relation. By the proof of Lemma 3 we can rewrite (8) as

$$\begin{aligned} q &= \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\alpha \in \mathcal{Z}_{\mathcal{A}}} q_\alpha - \sum_{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}}} q_{\alpha'} \right\} + q_0' \\ &= \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} q_\alpha + \sum_{\substack{\beta \in \mathcal{Z}_{\mathcal{A}} \\ 0<\beta<1}} q_\beta - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} q_{\alpha'} - \sum_{\substack{\beta \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ 0<\beta<1}} q_{\beta'} \right\} + q_0'. \end{aligned}$$

Lemma 5 and (14) imply that

$$\begin{aligned} \sum_{\substack{\beta \in \mathcal{Z}_{\mathcal{A}} \\ 0<\beta<1}} q_\beta &= \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} q_{(ST)^2\alpha'} \\ &= \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \left(q_{\alpha'} \mid ST - PP_0\left[q_{\alpha'}|ST\right]\right), \end{aligned}$$

and

$$\sum_{\substack{\beta\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ 0<\beta<1}} q_{\beta'} = \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} q_{(ST)^2\alpha} = \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} \left(q_\alpha \mid ST - PP_0\left[q_\alpha|ST\right]\right).$$

This gives us

$$q = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} q_\alpha \mid (I-ST) - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}} q_{\alpha'} \mid (I-ST) \right\} + q_0, \tag{15}$$

where

$$q_0 = q_0' - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}} PP_0\left[q_{\alpha'}|ST\right] + \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} PP_0\left[q_\alpha|ST\right]$$

is a rational function with a pole only at zero of order at most $2k$. The functions q_α and $q_{\alpha'}$ are given by (11) and (12).

The expression (15) emphasizes the fact that $q(z)$ satisfies the second relation (6) because each of the terms $q_\alpha \mid (I-ST)$ or $q_{\alpha'} \mid (I-ST)$ by itself satisfies the second relation. Since q must satisfy the second relation, q_0 must satisfy it as well. With this construction q_0 can be thought of as a correction term for the first relation (5). It is the sum of a rational period function of the form (7) and a function with a pole only at 0 which corrects for the possibility that $q - q_0$ (the rest of q) may not satisfy the first relation.

We will write $q(z)$ in a more explicit form. Let

$$q_{\alpha,l}(z) = \frac{1}{(z-\alpha)^l}, \text{ and}$$
$$q_{\alpha',l}(z) = \frac{1}{(z-\alpha')^l}. \tag{16}$$

Put $\beta' = (ST)^2\alpha$ and $\beta = (ST)^2\alpha'$. Then

$$(q_{\alpha,l} \mid ST)(z) = \frac{(\beta')^l}{z^{2k-l}(z-\beta')^l}, \text{ and}$$
$$(q_{\alpha',l} \mid ST)(z) = \frac{\beta^l}{z^{2k-l}(z-\beta)^l}. \tag{17}$$

With this notation (11) and (12) are

$$q_\alpha(z) = \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} q_{\alpha,l}, \tag{18}$$

and

$$q_{\alpha'}(z) = \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} q_{\alpha',l}. \tag{19}$$

Substituting (18) and (19) into (15) we have

$$\begin{aligned} q \;=\; & \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} q_{\alpha,l} \mid (I - ST) \\ & - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} q_{\alpha',l} \mid (I - ST) \Bigg\} + q_0, \end{aligned} \tag{20}$$

where q_0 may be written as

$$q_0(z) = \sum_{m=0}^{2k} \frac{b_m}{z^m} \tag{21}$$

with the $b_m, m = 0, 1, \ldots, 2k$ complex constants. Finally, if we use (16) and (17) to replace $q_{\alpha,l}$, $q_{\alpha,l}|ST$, $q_{\alpha',l}$ and $q_{\alpha',l}|ST$ then (20) is

$$\begin{aligned} q(z) = & \sum_{\mathcal{A}} C_{\mathcal{A}} \\ & \times \Bigg\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{1}{(z-\alpha)^l} - \frac{(\beta')^l}{z^{2k-l}(z-\beta')^l} \right) \\ & - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{1}{(z-\alpha')^l} - \frac{\beta^l}{z^{2k-l}(z-\beta)^l} \right) \Bigg\} \\ & + q_0(z). \end{aligned} \tag{22}$$

4 The direct Hecke theorem

In this section we prove that a modular integral of positive, even weight $2k$ on $\Gamma(1)$ leads to a Dirichlet series with a functional equation. We derive an explicit form for the remainder term in the functional equation, which is based on (20).

Suppose that $F(z)$ is an entire modular integral on $\Gamma(1)$ of weight $2k \in 2\mathbf{Z}^+$ with rational period function $q(z)$. We may assume without loss of generality that $F(z)$ is a *cusp* modular integral, *i.e.*, that $a_0 = 0$ in the Fourier expansion (1). This is because $F(z) \equiv -1$ is a trivial modular integral of weight $2k$ on $\Gamma(1)$ with rational period function $q(z) = 1 - z^{-2k}$.

Write $z = x + iy$ with $x, y \in \mathbf{R}$. It can be shown [7, 622-623] that F satisfies

$$|F(z)| \leq K \left(|z|^{\alpha} + y^{-\beta}\right), \quad z \in \mathcal{H} \tag{23}$$

for some positive real numbers K, α and β. It follows that the coefficients a_n in the Fourier expansion (1) for F satisfy

$$a_n = \mathcal{O}(n^{\beta}), \quad n \to +\infty. \tag{24}$$

This, with $a_0 = 0$ in (1), implies that

$$F(iy) = \mathcal{O}(e^{-2\pi y}), \quad y \to +\infty. \tag{25}$$

Because of (23) and (25) we may consider the *Mellin transform of* F,

$$\Phi(s) = \int_0^{\infty} F(iy) y^s \frac{dy}{y}, \tag{26}$$

a function of the complex variable $s = \sigma + it$. The integral in (26) converges for $\sigma > \beta$. For $\sigma > \beta + 1$, we can integrate term by term to get

$$\Phi(s) = (2\pi)^{-s} \Gamma(s) \phi(s), \tag{27}$$

where

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{28}$$

is the *Dirichlet series associated with* F. The bound on the growth of the coefficients a_n (24) implies that sum in (28) converges absolutely and uniformly on compact subsets of the right half plane $\sigma > \beta + 1$, so that $\phi(s)$ is analytic there.

Using the modular relation (4) we have

$$\begin{aligned}\int_0^1 F(iy) y^s \frac{dy}{y} &= \int_1^{\infty} F\left(\frac{-1}{iy}\right) y^{-s} \frac{dy}{y} \\ &= i^{2k} \int_1^{\infty} F(iy) y^{2k-s} \frac{dy}{y} + i^{2k} \int_1^{\infty} q(iy) y^{2k-s} \frac{dy}{y}.\end{aligned}$$

Thus

$$\Phi(s) = D(s) + E(s),$$

where

$$D(s) = \int_1^{\infty} F(iy)[y^s + i^{2k} y^{2k-s}] \frac{dy}{y} \tag{29}$$

and

$$E(s) = i^{2k} \int_1^\infty q(iy) y^{2k-s} \frac{dy}{y}. \tag{30}$$

It is not hard to see that $D(s)$ is entire and satisfies the functional equation

$$D(2k - s) - i^{2k} D(s) = 0. \tag{31}$$

From (20) and (21) we know that $q(z) = \mathcal{O}(1)$ as $|z| \to \infty$. Thus the integral defining $E(s)$ in (30) converges in the right half plane $\sigma > 2k$.

We will need more information about the analytic properties of $E(s)$, for which it is easiest to refer to [4]. Hawkins and Knopp prove in [4] that $E(s)$, and hence $\Phi(s)$, has a meromorphic continuation to the s-plane with, at worst, simple poles at integer points $m \le 2k$. They also show that $\Phi(s)$ is bounded in every lacunary vertical strip of the form

$$S(\sigma_1, \sigma_2; t_0): \ \sigma_1 \le \sigma \le \sigma_2, \ |t| \ge t_0 > 0, \tag{32}$$

where σ_1, σ_2, and t_0 are real numbers.

Since $\Phi(s)$ has a meromorphic continuation to the whole s-plane we may write the functional equation which is suggested by (31),

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s), \tag{33}$$

where $R(s)$ is a meromorphic function which we will call the *remainder term.* Then by (31) we have

$$R(s) = E(2k - s) - i^{2k} E(s), \tag{34}$$

from which it is clear that $R(s)$ depends only on the rational period function q and not on the modular integral F. The expression (34) (or (33)) implies that $R(s)$ satisfies the (first) relation

$$R(2k - s) + i^{2k} R(s) = 0, \tag{35}$$

which was first observed by Hawkins and Knopp [4].

We will find an explicit expression for $R(s)$ using (34) and the representation (20) for a rational period function on $\Gamma(1)$. This will give meaning to the functional equation (33) and it will enable us to prove a converse theorem. Put $E_a(s) = E(2k - s)$ and $E_b(s) = -i^{2k} E(s)$, so that

$$R(s) = E_a(s) + E_b(s). \tag{36}$$

By (30) we have

$$E_a(s) = i^{2k} \int_1^\infty q(iy) y^s \frac{dy}{y}, \tag{37}$$

and

$$E_b(s) = -\int_1^\infty q(iy) y^{2k-s} \frac{dy}{y}. \tag{38}$$

If we use the first relation (5) to replace $q(iy)$ in (38) we have

$$\begin{aligned} E_b(s) &= i^{2k} \int_1^\infty q\left(\frac{-1}{iy}\right) y^{-s} \frac{dy}{y} \\ &= i^{2k} \int_0^1 q(iy) y^s \frac{dy}{y}. \end{aligned} \tag{39}$$

It would be convenient to use (36), (37) and (39) to write $R(s)$ as the full Mellin transform of $q(z)$. We may not do this in general, however, since there is not necessarily any region in the s-plane in which both integrals converge. Yet the expression (36) is valid, since $E_a(s)$ and $E_b(s)$ are defined in terms of $E(s)$, which we know is meromorphic in the entire s-plane.

A simple calculation shows that the parts of $E_a(s)$ and $E_b(s)$ which arise from q_0 cancel each other, so that q_0 contributes nothing to the remainder term. As a result, we may write

$$R(s) = \hat{E}_a(s) + \hat{E}_b(s), \tag{40}$$

where

$$\hat{E}_a(s) = i^{2k} \int_1^\infty \{q(iy) - q_0(iy)\} y^s \frac{dy}{y},$$

and

$$\hat{E}_b(s) = i^{2k} \int_0^1 \{q(iy) - q_0(iy)\} y^s \frac{dy}{y}.$$

We will use (22) to write $q - q_0$ as the sum of two functions which we can consider separately. Let

$$\begin{aligned} q^{(1)}(z) &= \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{1}{(z-\alpha)^l} \right) \right. \\ &\quad \left. - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{1}{(z-\alpha')^l} \right) \right\}, \end{aligned} \tag{41}$$

and

$$q^{(2)}(z) = \sum_{\mathcal{A}} C_{\mathcal{A}} \left\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{-(\beta')^l}{z^{2k-l}(z-\beta')^l} \right) \right.$$

$$
- \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{-\beta^l}{z^{2k-l}(z-\beta)^l} \right) \Bigg\}, \tag{42}
$$

where the outer sum in each expression is on the equivalence classes which correspond to the irreducible pole sets of $q(z)$. Then by (22) we have

$$
q - q_0 = q^{(1)} + q^{(2)},
$$

so that

$$
\begin{aligned}
\hat{E}_a(s) &= i^{2k} \int_1^\infty q^{(1)}(iy) y^s \frac{dy}{y} + i^{2k} \int_1^\infty q^{(2)}(iy) y^s \frac{dy}{y} \\
&= \hat{E}_a^{(1)}(s) + \hat{E}_a^{(2)}(s),
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{E}_b(s) &= i^{2k} \int_0^1 q^{(1)}(iy) y^s \frac{dy}{y} + i^{2k} \int_0^1 q^{(2)}(iy) y^s \frac{dy}{y} \\
&= \hat{E}_b^{(1)}(s) + \hat{E}_b^{(2)}(s).
\end{aligned}
$$

This, with (40) gives us

$$
R(s) = \hat{E}_a^{(1)}(s) + \hat{E}_a^{(2)}(s) + \hat{E}_b^{(1)}(s) + \hat{E}_b^{(2)}(s). \tag{43}
$$

The integral for $\hat{E}_a^{(1)}(s)$ converges for $\sigma < 1$ and the integral for $\hat{E}_b^{(1)}(s)$ converges for $\sigma > 0$. Thus, for $0 < \sigma < 1$, we have

$$
\begin{aligned}
R^{(1)}(s) &= \hat{E}_a^{(1)}(s) + \hat{E}_b^{(1)}(s) \qquad (44) \\
&= i^{2k} \int_0^\infty q^{(1)}(iy) y^s \frac{dy}{y}. \qquad (45)
\end{aligned}
$$

The integral for $\hat{E}_a^{(2)}(s)$ converges for $\sigma < 2k$ and the integral for $\hat{E}_b^{(2)}(s)$ converges for $\sigma > 2k-1$. Thus, for $2k-1 < \sigma < 2k$, we have

$$
\begin{aligned}
R^{(2)}(s) &= \hat{E}_a^{(2)}(s) + \hat{E}_b^{(2)}(s) \qquad (46) \\
&= i^{2k} \int_0^\infty q^{(2)}(iy) y^s \frac{dy}{y}. \qquad (47)
\end{aligned}
$$

As a result the determination of $R(s)$ reduces to the evaluation of the integrals

$$
\begin{aligned}
R_{\alpha,l}^{(1)}(s) &= i^{2k} \int_0^\infty q_{\alpha,l}(iy) y^s \frac{dy}{y} = i^{2k} \int_0^\infty \frac{y^s}{(iy-\alpha)^l} \frac{dy}{y}, \\
R_{\alpha,l}^{(2)}(s) &= i^{2k} \int_0^\infty (q_{\alpha,l}|ST)(iy) y^s \frac{dy}{y} = i^{2k} (\beta')^l \int_0^\infty \frac{y^s}{(iy)^{2k-l}(iy-\beta')^l} \frac{dy}{y},
\end{aligned}
$$

$$\begin{aligned} R^{(1)}_{\alpha',l}(s) &= i^{2k}\int_0^\infty q_{\alpha',l}(iy)y^s\frac{dy}{y} = i^{2k}\int_0^\infty \frac{y^s}{(iy-\alpha')^l}\frac{dy}{y}, \text{ and} \\ R^{(2)}_{\alpha',l}(s) &= i^{2k}\int_0^\infty (q_{\alpha',l}|ST)(iy)y^s\frac{dy}{y} = i^{2k}\beta^l\int_0^\infty \frac{y^s}{(iy)^{2k-l}(iy-\beta)^l}\frac{dy}{y}, \end{aligned} \tag{48}$$

with $\beta = (ST)^2\alpha'$, $\beta' = (ST)^2\alpha$, and $1 \le l \le k$.

The evaluation of these integrals will involve exponential functions of the form $z^a = e^{a\log z}$, where $\log z = \log|z| + i\arg z$ for $z \in \mathbf{C}$. We will take the principal branch for each logarithm, using the convention that $-\pi \le \arg z < \pi$.

In order to evaluate the integrals in (48) we use the representation for the beta function [10, page 13]

$$B(a,b) = \int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}dt,$$

valid for Re $a > 0$ and Re $b > 0$. Let δ be a nonzero real number and change variables by putting $y = i\delta t$. If we use a contour integral to move the path of integration to the positive real axis, we have

$$B(a,b) = i^b\delta^b\int_0^\infty \frac{y^a}{(y+i\delta)^{a+b}}\frac{dy}{y}.$$

We replace b with $b-a$ and rearrange to get

$$\int_0^\infty \frac{y^a}{(iy-\delta)^b}\frac{dy}{y} = i^{a-2b}\delta^{a-b}B(a,b-a), \tag{49}$$

for $0 < \text{Re } a < \text{Re } b$, $\delta \in \mathbf{R}$, $\delta \ne 0$.

Using (49) to evaluate the integrals in (48), we have

$$\begin{aligned} R^{(1)}_{\alpha,l}(s) &= i^{s+2k-2l}\alpha^{s-l}B(s,l-s), \\ R^{(2)}_{\alpha,l}(s) &= i^{s-2k}(\beta')^{s-2k+l}B(s-2k+l,2k-s), \\ R^{(1)}_{\alpha',l}(s) &= i^{s+2k-2l}(\alpha')^{s-l}B(s,l-s), \\ \text{and } R^{(2)}_{\alpha',l}(s) &= i^{s-2k}\beta^{s-2k+l}B(s-2k+l,2k-s). \end{aligned} \tag{50}$$

Now we may use (44) and (46) in (43) to write

$$R(s) = R^{(1)}(s) + R^{(2)}(s). \tag{51}$$

The expressions (45) and (41), along with the notation of (48), imply that

$$R^{(1)} = \sum_{\mathcal{A}} C_{\mathcal{A}}\left\{\sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}(\alpha-\alpha')^{l-k}(-1)^{l-k}R^{(1)}_{\alpha,l}\right.$$

$$
- \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} R^{(1)}_{\alpha',l} \Bigg\} .
$$

Similarly, (47) and (42) imply that

$$
\begin{aligned}
R^{(2)} \;=\; & \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(-R^{(2)}_{\alpha,l}\right) \\
& - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(-R^{(2)}_{\alpha',l}\right) \Bigg\} .
\end{aligned}
$$

Using these expressions in (51) we have

$$
\begin{aligned}
R \;=\; & \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} (R^{(1)}_{\alpha,l} - R^{(2)}_{\alpha,l}) \\
& - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (R^{(1)}_{\alpha',l} - R^{(2)}_{\alpha',l}) \Bigg\} .
\end{aligned} \tag{52}
$$

Then, by (50), we have the expression

$$
\begin{aligned}
R(s) \;=\; & \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \\
& \times (i^{s+2k-2l} \alpha^{s-l} B(s, l-s) - i^{s-2k} (\beta')^{s-2k+l} B(s-2k+l, 2k-s)) \\
& - \sum_{\substack{\alpha \in \mathcal{Z}_{\theta \mathcal{A}^{-1}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \\
& \times (i^{s+2k-2l} (\alpha')^{s-l} B(s, l-s) - i^{s-2k} \beta^{s-2k+l} B(s-2k+l, 2k-s)) \Bigg\} .
\end{aligned} \tag{53}
$$

We can replace β and β', using $\beta = (ST)^2\alpha' = \frac{-1}{\alpha'-1}$ and $\beta' = (ST)^2\alpha = \frac{-1}{\alpha-1}$. After simplifying, the result is

$$\begin{aligned} R(s) \;=\; & \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k} \\ & \qquad \times \left(i^s\alpha^{s-l}B(s,l-s) - i^{-s}(\alpha-1)^{2k-s-l}B(s-2k+l,2k-s)\right) \\ & - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} \\ & \qquad \times \left(i^s(\alpha')^{s-l}B(s,l-s) - i^{-s}(\alpha'-1)^{2k-s-l}B(s-2k+l,2k-s)\right) \Bigg\}. \end{aligned} \tag{54}$$

We have proved the following theorem.

Theorem 6 *Suppose that $F(z)$ is an entire modular integral of weight $2k \in 2\mathbf{Z}^+$ on $\Gamma(1)$, with rational period function $q(z)$ given by (22); suppose that F has the Fourier expansion (1) with zero constant term, so that (25) holds. Let $\Phi(s)$ be defined by (26) for $\sigma > \beta$.*

Then for $\sigma > \beta + 1$, $\Phi(s)$ is also given by (27) and (28), and
(a) $\Phi(s)$ has a meromorphic continuation to the whole s-plane with, at worst, simple poles at integer points $m \le 2k$. $\Phi(s)$ is represented for $\sigma > \beta$ by

$$\Phi(s) = D(s) + E(s).$$

$D(s)$ is given by (29) and is entire, and $E(s)$ is given by (30) and has a meromorphic continuation to the whole s-plane. Furthermore,
(b) $\Phi(s)$ is bounded in every lacunary vertical strip of the form (32), and
(c) $\Phi(s)$ satisfies the functional equation (33) where $R(s)$ is given by (54).

Remark. The expression (54) for $R(s)$ involves exponential functions and beta functions, as does the expression Hawkins and Knopp obtain in [4] for a remainder term associated with Γ_θ. Hawkins and Knopp get their result by calculating $E(s)$ directly, using integrals from 1 to ∞ as in (30). These partial Mellin transforms involve ordinary hypergeometric functions. Hawkins and Knopp calculate the remainder term using (34) and several hypergeometric function identities, and in the end the hypergeometric functions drop out. We have bypassed the steps involving hypergeometric functions by using the first relation (5) to combine integrals as in (45) and (47). The resulting expressions, which are full Mellin transforms, involve beta functions.

5 The second relation

We have already observed that the remainder term $R(s)$ satisfies the relation (35). In this section we will describe a second relation which $R(s)$ must satisfy. This second relation follows from the fact that the corresponding rational period function $q(z)$ satisfies (6).

Suppose that α is one of the poles of $q(z)$ which is denoted by α or α' in (22). (In order to simplify the notation we will suppress the prime if α is negative.) Let $\beta = (ST)^2\alpha$ and $\gamma = (ST)^2\beta$, so that $\alpha = (ST)^2\gamma$. Then β represents one of the poles of $q(z)$ which is denoted by β or β' in (22). Fix l, $1 \le l \le k$ and put

$$\begin{aligned} q_1(z) &= \frac{1}{(z-\alpha)^l}, \\ q_2(z) &= (q_1|ST)(z) = \frac{\beta^l}{z^{2k-l}(z-\beta)^l}, \\ q_3(z) &= (q_1|(ST)^2)(z) = \frac{(\gamma-1)^l}{(z-1)^{2k-l}(z-\gamma)^l}. \end{aligned} \tag{55}$$

Then by (22) $q - q_0$ is a linear combination of terms of the form

$$\hat{q} = q_1 - q_2,$$

and $|ST$ maps $q_1 \to q_2 \to q_3 \to q_1$. As a result each term $\hat{q}$ satisfies the second relation (6), since

$$\hat{q}|ST = q_2 - q_3$$

and

$$\hat{q}|(ST)^2 = q_3 - q_1.$$

In order to write the terms of $R(s)$ which correspond to q_1, q_2 and q_3 we put

$$R_j(s) = i^{2k}\int_0^\infty q_j(iy)y^s\,\frac{dy}{y} \tag{56}$$

for $j = 1, 2, 3$. Then $R(s)$ is a linear combination of terms of the form

$$\hat{R} = R_1 - R_2.$$

The integral defining $R_1(s)$ converges for $0 < \sigma < l$, the one defining $R_2(s)$ converges for $2k - l < \sigma < 2k$, and the one defining $R_3(s)$ converges for $0 < \sigma < 2k$. We know that R_1 and R_2 are in fact meromorphic in the entire s-plane, since they have been written explicitly in (50). $R_3(s)$ has a meromorphic continuation to the entire s-plane as well, but we shall not need that result for our proof. Since the integral defining $R_3(s)$ converges in the strip $0 < \sigma < 2k$, we have $R_1(s)$, $R_2(s)$ and $R_3(s)$ all defined in the strip $0 < \sigma < 2k$.

Let a, b and ϕ be real numbers, let r, m and l be nonnegative integers, and put

$$R_{r,m,l}(s;a,b,\phi) = i^{2k}\int_0^\infty \frac{y^s}{(iy-a)^r(iy-b)^m(iy-\phi)^l}\frac{dy}{y}. \tag{57}$$

The parameters a, b and ϕ will represent poles of terms of the rational period function, and r, m and l will denote the respective orders of the poles. The region of convergence for the integral in (57) depends on the values of a, b, ϕ, r, m, and l. With this notation we have

$$\begin{aligned} R_1(s) &= R_{0,0,l}(s;0,1,\alpha), \\ R_2(s) &= \beta^l R_{2k-l,0,l}(s;0,1,\beta) \\ \text{and }\ R_3(s) &= (\gamma-1)^l R_{0,2k-l,l}(s;0,1,\gamma), \end{aligned} \tag{58}$$

with $\beta = (ST)^2\alpha$, $\gamma = (ST)^2\beta$ and $\alpha = (ST)^2\gamma$.

Define the mapping ρ by

$$\rho(R_{r,m,l}(s;0,1,\phi)) = i^{2k}R_{r,m,l}(2k-s;-1,0,\phi-1). \tag{59}$$

It is clear that ρ is linear, *i.e.*, that

$$\rho(a_1R_1 + a_2R_2) = a_1\rho(R_1) + a_2\rho(R_2)$$

for any constants a_1 and a_2 and functions R_1 and R_2 of the form (57). The mapping ρ is the image of the mapping $|ST$ on the modular integral side of the correspondence.

The next lemma will allow us to rewrite the right hand side of (59).

Lemma 7

$$R_{r,m,l}(2k-s;-1,0,\phi-1) = i^{-2k-2m}((ST)^2\phi)^l R_{2k-r-m-l,r,l}(s;0,1,(ST)^2\phi)$$

The proof uses the integral definition (57), a change of variables, and some simple manipulations.

Using Lemma 7 we may write the mapping ρ in an alternative way as

$$\rho(R_{r,m,l}(s;0,1,\phi)) = i^{-2m}((ST)^2\phi)^l R_{2k-r-m-l,r,l}(s;0,1,(ST)^2\phi). \tag{60}$$

We can now use ρ to state a relation which $\hat{R}$ satisfies and which reflects the fact that $\hat{q}$ satisfies (6).

Theorem 8 *Let R_1, R_2 and R_3 be given by (56) and (55). Suppose that $\hat{R} = R_1 - R_2$, and suppose that ρ is the mapping defined by (59). Then $\hat{R}$ satisfies the relation*

$$\hat{R} + \rho(\hat{R}) + \rho^2(\hat{R}) = 0.$$

Proof: We first show that $\rho : R_1 \to R_2 \to R_3 \to R_1$. Using (60) we have

$$\begin{aligned} \rho(R_1(s)) &= \rho(R_{0,0,l}(s; 0, 1, \alpha)) \\ &= \beta^l R_{2k-l,0,l}(s; 0, 1, \beta) \\ &= R_2(s). \end{aligned}$$

Also,

$$\begin{aligned} \rho(R_2(s)) &= \beta^l \rho(R_{2k-l,0,l}(s; 0, 1, \beta)) \\ &= \beta^l \gamma^l R_{0,2k-l,l}(s; 0, 1, \gamma) \\ &= (\gamma - 1)^l R_{0,2k-l,l}(s; 0, 1, \gamma) \\ &= R_3(s), \end{aligned}$$

since $\beta = ST\gamma = \frac{\gamma-1}{\gamma}$. Finally,

$$\begin{aligned} \rho(R_3(s)) &= (\gamma - 1)^l \rho(R_{0,2k-l,l}(s; 0, 1, \gamma)) \\ &= (\gamma - 1)^l \alpha^l i^{2l} R_{0,0,l}(s; 0, 1, \alpha) \\ &= R_{0,0,l}(s; 0, 1, \alpha) \\ &= R_1(s), \end{aligned}$$

since $\gamma = ST\alpha = 1 - 1/\alpha$, or $\gamma - 1 = -1/\alpha$.

Thus we have

$$\rho(\hat{R}) = R_2 - R_3$$

and

$$\rho^2(\hat{R}) = R_3 - R_1,$$

so that

$$\hat{R} + \rho(\hat{R}) + \rho^2(\hat{R}) = 0. \qquad \square$$

We can now extend the second relation to the entire remainder term.

Corollary 9 *Suppose that $R(s)$ is the remainder term for a Dirichlet series which corresponds to an entire modular integral F with rational period function q on $\Gamma(1)$. Then $R(s)$ satisfies*

$$R + \rho(R) + \rho^2(R) = 0, \tag{61}$$

where ρ is the mapping (59).

Proof: Any rational period function on $\Gamma(1)$ can be written as in (20) so that $q - q_0$ is a linear combination of terms of the form $\hat{q} = q_1 - q_2$ with q_1 and q_2 given by (55). Then, since q_0 makes no contribution to the remainder term, $R(s)$ is a linear combination of terms of the form $\hat{R} = R_1 - R_2$ where R_1 and R_2 are given by (56), hence by (58). Theorem 8 implies that $\hat{R}$ satisfies (61). The corollary follows from the fact that ρ is a linear map. $\square$

6 The converse Hecke theorem

We now prove the following converse to Theorem 6.

Theorem 10 *Suppose the Dirichlet series*

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \tag{62}$$

converges absolutely in the half-plane $\sigma > \gamma$*; suppose that the function* $\Phi(s)$ *defined by*

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s) \tag{63}$$

satisfies:
(a) $\Phi(s)$ *has a meromorphic continuation to the whole s-plane with, at worst, simple poles at integer points s;*
(b) $\Phi(s)$ *is bounded in every lacunary vertical strip of the form*

$$S(\sigma_1, \sigma_2; t_0):\ \sigma_1 \le \sigma \le \sigma_2,\ |t| \ge t_0 > 0; \quad \textit{and} \tag{64}$$

(c) $\Phi(s)$ *satisfies the functional equation*

$$\Phi(2k - s) - i^{2k}\Phi(s) = R(s), \tag{65}$$

where $R(s)$ *is given by*

$$\begin{aligned} R(s) \;=\; & \sum_{\mathcal{A}} C_{\mathcal{A}} \Bigg\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} \\ & \quad \times (i^{s+2k-2l}\alpha^{s-l}B(s,l-s) - i^{s-2k}(\beta')^{s-2k+l}B(s-2k+l,2k-s)) \\ & - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k} \\ & \quad \times (i^{s+2k-2l}(\alpha')^{s-l}B(s,l-s) - i^{s-2k}\beta^{s-2k+l}B(s-2k+l,2k-s)) \Bigg\}. \end{aligned} \tag{66}$$

with the outer sum on a finite number of equivalence classes of binary quadratic forms, $\beta = (ST)^2\alpha'$*, and* $\beta' = (ST)^2\alpha$*.*

Then $\phi(s)$ is the Dirichlet series associated with an entire modular integral F of weight $2k$ on $\Gamma(1)$ with rational period function

$$q(z) = \sum_{\mathcal{A}} C_{\mathcal{A}}$$
$$\times \left\{ \sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k}(-1)^{l-k} \left(\frac{1}{(z-\alpha)^l} - \frac{(\beta')^l}{z^{2k-l}(z-\beta')^l} \right) \right.$$
$$\left. - \sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha-\alpha')^{l-k} \left(\frac{1}{(z-\alpha')^l} - \frac{\beta^l}{z^{2k-l}(z-\beta)^l} \right) \right\}$$
$$+q_0(z) \tag{67}$$

where q_0 has a pole only at 0 of order at most $2k$.

(For convenience we have renumbered (33), (53) and (22) as (65), (66) and (67), respectively.)

Proof: Since $\phi(s)$ converges absolutely in $\sigma > \gamma$, we have

$$a_n = \mathcal{O}(n^{\gamma-1}).$$

Thus we may form

$$F(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z},$$

which converges for all $z = x + iy$ in the upper half-plane $\mathcal{H}$. Now e^{-y} is the inverse Mellin transform of the gamma function $\Gamma(s)$, *i.e.*,

$$e^{-y} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) y^{-s} ds \tag{68}$$

for any positive real numbers c and y. Let L be a positive integer with $L > \gamma$ and $L \geq 2k$. Fix c with $L < c < L+1$, from which it follows that for $\sigma \geq c$,

$$|\phi(s)| \leq \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} < \infty.$$

Then for $y > 0$,

$$\begin{aligned} F(iy) &= \sum_{n=1}^{\infty} a_n \left(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)(2\pi n y)^{-s} ds \right) \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (2\pi)^{-s}\Gamma(s) \left(\sum_{n=1}^{\infty} \frac{a_n}{n^s} \right) y^{-s} ds \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Phi(s) y^{-s} ds, \end{aligned} \tag{69}$$

and $F(iy)$ is the inverse Mellin transform of $\Phi(s)$. The interchange of the sum and integral above is valid by Stirling's Formula,

$$|\Gamma(\sigma+it)| \sim \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}, \quad |t| \to \infty. \tag{70}$$

We would like to move the line of integration in (69) from $\sigma = c$ to $\sigma = 2k-c$. In order to do this we note that by Stirling's Formula (70) and the fact that $\phi(s)$ is bounded on the line $\sigma = c$,

$$\Phi(c+it) = \mathcal{O}\left(e^{-\epsilon|t|}\right), \quad |t| \to \infty,$$

for any ϵ, $0 < \epsilon < \pi/2$. By the functional equation (65) and the expression for $R(s)$ (66) we have that

$$\Phi(2k-c-it) = \mathcal{O}\left(e^{-\epsilon|t|}\right), \quad |t| \to \infty.$$

By assumption, $\Phi(s)$ is bounded in the lacunary vertical strip $S(2k-c, c; t_0)$ for any $t_0 > 0$. Thus the Phragmén-Lindelöf principle implies that

$$\Phi(s) = \mathcal{O}\left(e^{-\epsilon|t|}\right), \quad |t| \to \infty,$$

uniformly in $2k - c \leq \sigma \leq c$. Now y^{-s} is bounded uniformly in $2k - c \leq \sigma \leq c$, so

$$\lim_{|T| \to \infty} \int_{2k-c+iT}^{c+iT} \Phi(s)y^{-s}ds = 0. \tag{71}$$

Using (71) we may move the line of integration from $\sigma = c$ to $\sigma = 2k - c$. Since $L < c < L+1$ and $2k - L - 1 < 2k - c < 2k - L$, we pick up the residues of the integrand at $s = 2k - L, 2k - L - 1, \ldots, L - 1$. (The integrand $\Phi(s)y^{-s}$ does not have a pole at $s = L$, since $\phi(s)$ converges absolutely for $\sigma = L$.) We get

$$F(iy) = \frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} \Phi(s)y^{-s}ds + \frac{1}{2\pi i}\sum_{m=2k-L}^{L-1} \operatorname*{Res}_{s=m}\{\Phi(s)y^{-s}\}.$$

Using the functional equation (65) we have

$$\begin{aligned} F(iy) &= \frac{i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} \Phi(2k-s)y^{-s}ds - \frac{i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} R(s)y^{-s}ds \\ &\quad + \frac{1}{2\pi i}\sum_{m=2k-L}^{L-1} \operatorname*{Res}_{s=m}\{\Phi(s)y^{-s}\}. \end{aligned} \tag{72}$$

A change of variables and (69) imply that

$$\frac{i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} \Phi(2k-s)y^{-s}ds = (F|T)(iy).$$

Thus (72) is

$$(F|T)(iy) - F(iy) = \frac{i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} R(s)y^{-s}ds - \frac{1}{2\pi i}\sum_{m=2k-L}^{L-1} \operatorname*{Res}_{s=m}\{\Phi(s)y^{-s}\}. \tag{73}$$

We will show that the right hand side of (73) is $q(iy)$, with $q(z)$ given by (67). The sum in (73) is $q_0^{(a)}(iy)$, the rational function given by

$$q_0^{(a)}(z) = \sum_{m=2k-L}^{L-1} c_m z^{-m}, \tag{74}$$

where

$$c_m = \frac{i^m}{2\pi i}\operatorname*{Res}_{s=m}\{\Phi(s)\}.$$

We next evaluate the integral in (73). Because of the expression (66) for $R(s)$ we need to evaluate the integrals

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} i^{s+2k-2l}\delta^{s-l}B(s, l-s)y^{-s}ds, \tag{75}$$

and

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} i^{s-2k}\delta^{s-2k+l}B(s-2k+l, 2k-s)y^{-s}ds, \tag{76}$$

for $l \in \mathbf{Z}$, $1 \le l \le k$, and $\delta \in \mathbf{R}$, $\delta \ne 0$.

If we let $a = s$ and $b = l$ in (49) we have

$$\int_0^\infty \frac{y^s}{(iy-\delta)^l}\frac{dy}{y} = i^{s-2l}\delta^{s-l}B(s, l-s),$$

for $0 < \sigma < l$ and $\delta \in \mathbf{R}, \delta \ne 0$. Since $\frac{1}{(iy-\delta)^l}$ is of bounded variation, we have the inverse Mellin transform [12, Theorem 9a]

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} i^{s-2l}\delta^{s-l}B(s, l-s)y^{-s}ds = \frac{1}{(iy-\delta)^l},$$

for $0 < c < l$ and $y > 0$. Letting $c = 1/2$ we have

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} i^{s+2k-2l}\delta^{s-l}B(s, l-s)y^{-s}ds = \frac{i^{2k}}{(iy-\delta)^l}, \tag{77}$$

for any $y > 0$. To evaluate (75), we move the line of integration from $\sigma = 2k-c$ to $\sigma = 1/2$, which we may do because $B(s, l-s)$ decays exponentially on any vertical line. We pick up the negatives of the residues at $s = 2k-L, 2k-L+$

$1, \ldots, 0$, since $2k - L - 1 < 2k - c < 2k - L$ and we are moving the line to the right. Then we apply (77) to get

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} i^{s+2k-2l}\delta^{s-l}B(s, l-s)y^{-s}ds \\
&= \frac{i^{2k}}{(iy-\delta)^l} - \frac{1}{2\pi i}\sum_{m=2k-L}^{0} \operatorname*{Res}_{s=m}\{i^{s+2k-2l}\delta^{s-l}B(s, l-s)y^{-s}\} \\
&= \frac{i^{2k}}{(iy-\delta)^l} - \sum_{m=2k-L}^{0} \frac{a_m(\delta, l)}{(iy)^m}. \qquad (78)
\end{aligned}$$

The coefficients $a_m(\delta, l)$ are given by

$$\begin{aligned}
a_m(\delta, l) &= \frac{i^{2m+2k-2l}\delta^{m-l}}{2\pi i}\operatorname*{Res}_{s=m}\{B(s, l-s)\} \\
&= \frac{(-1)^{l-k}\delta^{m-l}\Gamma(l-m)}{2\pi i\Gamma(1-m)\Gamma(l)}, \qquad (79)
\end{aligned}$$

for $1 \le l \le k$ and $m = 2k - L, 2k - L + 1, \ldots, 0$.

To evaluate (76) we observe that

$$\begin{aligned}
\int_0^\infty \frac{y^s}{(iy)^{2k-l}(iy-\delta)^l}\frac{dy}{y} &= i^{l-2k}\int_0^\infty \frac{y^{s-2k+l}}{(iy-\delta)^l}\frac{dy}{y} \\
&= i^s\delta^{s-2k}B(s-2k+l, 2k-s),
\end{aligned}$$

for $2k - l < \sigma < 2k$, $1 \le l \le k$, using (49) with $a = s - 2k + l$ and $b = l$. Since $\frac{1}{(iy)^{2k-l}(iy-\delta)^l}$ is of bounded variation, we have the inverse Mellin transform [12, Theorem 9a],

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} i^s\delta^{s-2k}B(s-2k+l, 2k-s)y^{-s}ds = \frac{1}{(iy)^{2k-l}(iy-\delta)^l}$$

for $2k - l < c < 2k$ and $y > 0$. In particular, letting $c = 2k - 1/2$ we have

$$\frac{1}{2\pi i}\int_{2k-1/2-i\infty}^{2k-1/2+i\infty} i^{s-2k}\delta^{s-2k+l}B(s-2k+l, 2k-s)y^{-s}ds = \frac{i^{-2k}\delta^l}{(iy)^{2k-l}(iy-\delta)^l} \qquad (80)$$

for any $y > 0$. Then to evaluate (76), we move the line of integration from $\sigma = 2k - c$ to $\sigma = 2k - 1/2$, which we may do because $B(s - 2k + l, 2k - s)$ decays exponentially on any vertical line. We pick up the negatives of the residues at $s = 2k - L, 2k - L + 1, \ldots, 2k - 1$, since $2k - L - 1 < 2k - c < 2k - L$ and we are moving the line to the right. Then we apply (80) to get

$$\frac{1}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} i^{s-2k}\delta^{s-2k+l}B(s-2k+l, 2k-s)y^{-s}ds$$

$$= \frac{i^{-2k}\delta^l}{(iy)^{2k-l}(iy-\delta)^l} - \frac{1}{2\pi i}\sum_{m=2k-L}^{2k-1} \operatorname*{Res}_{s=m}\{i^{s-2k}\delta^{s-2k+l}B(s-2k+l,2k-s)y^{-s}\}$$

$$= \frac{i^{-2k}\delta^l}{(iy)^{2k-l}(iy-\delta)^l} - \sum_{m=2k-L}^{2k-l}\frac{b_m(\delta,l)}{(iy)^m}. \tag{81}$$

The coefficients $b_m(\delta,l)$ are given by

$$\begin{aligned} b_m(\delta,l) &= \frac{i^{2m-2k}\delta^{m-2k+l}}{2\pi i}\operatorname*{Res}_{s=m}\{B(s-2k+l,2k-s)\} \\ &= \frac{(-1)^{k+l}\delta^{m-2k+l}}{2\pi i(2k-m-l)!}, \end{aligned} \tag{82}$$

for $1 \le l \le k$ and $m = 2k-L, 2k-L+1, \ldots, 2k-l$.

We can now evaluate the integral in (73). If we substitute (66) and use (78) and (81) we get

$$\begin{aligned} &\frac{i^{-2k}}{2\pi i}\int_{2k-c-i\infty}^{2k-c+i\infty} R(s)y^{-s}ds = \sum_{\mathcal{A}} C_{\mathcal{A}} \\ &\quad\times\Bigg\{\sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}(\alpha-\alpha')^{l-k}(-1)^{l-k}\left(\frac{1}{(iy-\alpha)^l} - \frac{(\beta')^l}{(iy)^{2k-l}(iy-\beta')^l}\right) \\ &\quad -\sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}(\alpha-\alpha')^{l-k}\left(\frac{1}{(iy-\alpha')^l} - \frac{\beta^l}{(iy)^{2k-l}(iy-\beta)^l}\right)\Bigg\} \\ &\quad - q_0^{(b)}(iy), \end{aligned} \tag{83}$$

where $q_0^{(b)}(z)$ is the rational function

$$\begin{aligned} &q_0^{(b)}(z) \\ &\quad = i^{-2k}\sum_{\mathcal{A}} C_{\mathcal{A}}\Bigg\{\sum_{\substack{\alpha\in\mathcal{Z}_{\mathcal{A}}\\ \alpha>1}}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}(\alpha-\alpha')^{l-k}(-1)^{l-k} \\ &\qquad\qquad \times\left(\sum_{m=2k-L}^{0}\frac{a_m(\alpha,l)}{z^m} - \sum_{m=2k-L}^{2k-l}\frac{b_m(\beta',l)}{z^m}\right) \\ &\qquad -\sum_{\substack{\alpha\in\mathcal{Z}_{\theta\mathcal{A}^{-1}}\\ \alpha>1}}\sum_{l=1}^{k}\binom{2k-1-l}{k-l}(\alpha-\alpha')^{l-k} \end{aligned}$$

$$\times \left(\sum_{m=2k-L}^{0} \frac{a_m(\alpha', l)}{z^m} - \sum_{m=2k-L}^{2k-l} \frac{b_m(\beta, l)}{z^m} \right) \Bigg\}. \quad (84)$$

Using (74), (83) and (84) in the right hand side of (73) we have, for $y > 0$, that

$$(F|T)(iy) = F(iy) + q(iy), \quad (85)$$

where

$$\begin{aligned} q(z) &= \sum_{\mathcal{A}} C_{\mathcal{A}} \\ &\times \Bigg\{ \sum_{\substack{\alpha \in \mathcal{Z}_{\mathcal{A}} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} (-1)^{l-k} \left(\frac{1}{(z-\alpha)^l} - \frac{(\beta')^l}{z^{2k-l}(z-\beta')^l} \right) \\ &- \sum_{\substack{\alpha \in \mathcal{Z}_{\theta\mathcal{A}-1} \\ \alpha > 1}} \sum_{l=1}^{k} \binom{2k-1-l}{k-l} (\alpha - \alpha')^{l-k} \left(\frac{1}{(z-\alpha')^l} - \frac{\beta^l}{z^{2k-l}(z-\beta)^l} \right) \Bigg\} \\ &+ \tilde{q}_0(z) \end{aligned} \quad (86)$$

and

$$\begin{aligned} \tilde{q}_0(z) &= -q_0^{(a)} - q_0^{(b)} \\ &= \sum_{m=2k-L}^{L-1} \frac{d_m}{z^m}. \end{aligned} \quad (87)$$

The expression given by (86) and (87) for q is the same as (67) except that $\tilde{q}_0$ may have a pole at ∞ or a pole at 0 of order greater than $2k$. We use the identity theorem to extend (85) to

$$(F|T)(z) = F(z) + q(z)$$

for $z \in \mathcal{H}$. Thus $F(z)$ is a modular integral of weight $2k$ on $\Gamma(1)$ with rational period function $q(z)$. Since $q(z)$ is a rational period function of positive weight $2k$ for $\Gamma(1)$, it cannot have a pole at ∞ and the pole at zero is of order at most $2k$ [3]. Thus we must have $d_m = 0$ for $2k - L \leq m < 0$ and for $2k < m \leq L - 1$ in (87). As a result $\tilde{q}_0$ has the form

$$\tilde{q}_0(z) = \sum_{m=0}^{2k} \frac{d_m}{z^m} \quad (88)$$

and $q(z)$ is the rational period function in (67). □

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