NOTES ON THE FUNDAMENTAL THEOREM OF CALCULUS

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Here we will try to understand both the meaning and the proof of the Fundamental Theorem of Calculus. Let me say at the outset: Focusing on proofs is **not** what we'll be doing all semester long. But this theorem deserves careful thought, and at least once, I think I should give you a glimpse of what more theoretical mathematics is about.

The Fundamental Theorem of Calculus (FTC) reveals a deep relationship between integrals and derivatives. We've already spent considerable time discussing what an integral really is. (And don't you forget it!) Let's quickly review derivatives:

The derivative f' of a function f can be thought of as measuring:

- The rate of change of f: If we change x by a small value Δx, then $f(x+\Delta x)-f(x)=\Delta y$ is approximately $f'(x)\Delta x$. As $\Delta x \to 0$, the approximation becomes more precise. Thus $f'(x)$ shows how fast y changes in comparison to x.
- The slope of a line tangent to the graph of f at $(x, f(x))$. In fact, when you think about it, it is not so easy to say exactly what we mean by a tangent line – although we all "know it when we see it." The derivative really serves to make this notion precise.

  A *secant line* is a line which passes through a curve at two (nearby) points. Once we have two points, we can use the slope formula

  $$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

  If the two x-values are x and $x + \Delta x$, then the slope of the secant line becomes

  $$m = \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

  By drawing the two points defining the secant line closer and closer together – in other words, by making Δx smaller and smaller – the secant lines approach the tangent line. So

  $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

  gives the slope of the tangent line. This is where the definition of the derivative,

  $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

  comes from.

This is a good place to mention a theorem about derivatives which we will need later on:

Theorem 0.1 (Mean Value Theorem)**.** *Suppose f is a differentiable function defined on an interval $[c,d]$. Then there exists a number $x^* \in [c,d]$ such that*

$$f'(x^*)(d-c) = f(d) - f(c).$$

We can understand the statement better if we do a bit of algebra on the conclusion: Divide both sides by $d-c$. Then it says there exists x^* such that $f'(x^*) = \frac{f(d)-f(c)}{d-c}$. But $\frac{f(d)-f(c)}{d-c}$ is the slope of the secant line through $(c, f(c))$ and $(d, f(d))$. It is also the average (or *mean*) rate of change of f over the entire interval $[c,d]$. So the theorem says that there is a particular point x^* where the instantaneous rate of change is the same as the average rate of change over the whole interval – or where the slope of the tangent line is equal to that of the beginning-to-end secant line.

We'll get back to this. But now (Drum roll, please....)

Theorem 0.2 (Fundamental Theorem of Calculus)**.** *If f is a continuous function on a closed interval $[a,b]$, and F is any function such that $F' = f$ (an "antiderivative" of f), then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

That is, **we can use what we know about DERIVATIVES to calculate INTEGRALS.** More precisely, if we start with a function f and can figure out how to *reverse* the process of calculating its derivative, then we can use the resulting antiderivative to calculate $\int_a^b f(x)dx$ – even though what $\int_a^b f(x)dx$ actually means is something completely different. (Remember – we spent three or four class days building up the definition of integrals in terms of Riemann sums!)

Example. Calculate $\int_0^1 (x^2 - x^3)dx$ exactly. (Recall that we approximated this integral using left sums in an exercise in class.)

We need to find a function F so that $F' = f$. Verify that $F(x) = \frac{x^3}{3} - \frac{x^4}{4} + 17$ has the right derivative. (Just make sure it works. Where it comes from is another matter, which we'll discuss in chapter 6).

Then by the Theorem,

$$\int_0^1 (x^2 - x^3)dx = F(1) - F(0) = (\frac{1}{3} - \frac{1}{4} + 17) - (0 - 0 + 17) = \frac{1}{12}.$$

Clearly, this is much nicer than trying to calculate a limit of Riemann sums.

Your text sketches some of the reasoning behind the FTC, but leaves out some details to make it more readable. Here is my suggestion: Read their argument, then read this one through once, lightly. Then take a study break and watch Survivor XXIV or something. Then come back to the proof of the theorem (below), and work backwards – each time an earlier result is referred to, try to understand what role it plays in the argument. Understanding the structure of an argument like this takes time and practice.

We begin with some fundamental results:

Lemma 0.3. *Let f be a continuous function on a closed interval $[c,d] \subseteq \mathbb{R}$. Then there exist numbers $s, l \in [c,d]$ such that*

- *$f(s) \leq f(x)$ for all $x \in [c,d]$, and*
- *$f(l) \geq f(x)$ for all $x \in [c,d]$.*

(That is, the smallest value of the function occurs at s, and the largest, at l.)

A shorter way to say this is that every continuous function on a closed interval *attains* its minimum and maximum.

At first, this might seem obvious. Consider the following examples to see why the hypotheses are necessary.

Example.

- The function $f(x) = 1/x$ is continuous on the *open* interval $(0, 1)$, but such s and l do not exist. Why not?
- Consider the function
$$f(x) = \begin{cases} 0 & \text{if } x = \pi/2, \\ \sin(x) & \text{if } 0 \leq x \leq \pi, \text{ but } x \neq \frac{\pi}{2}. \end{cases}$$
It is defined on the closed interval $[0, \pi]$, but is not quite continuous, and the place where the maximum "ought" to be has been tampered with by some psychotic math teacher. So there is *no* maximum value: If you give me any candidate for l near $x = \pi$, (like $\pi - .001$), I can find another point l' (like $\pi - .0001$) where $f(l') > f(l)$.

The proof of this lemma requires a lot of foundational work, and in particular, a precise definition of the set of real numbers, starting from scratch (or at least, starting from the set of integers $\mathbb{Z}$). We won't get into that here.

But we can use this theorem to justify a key ingredient in an"improved" definition of the definite integral. Recall that our provisional definition was in terms of limits of left and right Riemann sums.

Setup for improved definition: Suppose f is a continuous function defined on a closed interval $[a, b]$. As before, partition $[a, b]$ into n equal parts, and label the dividing points $a = t_0, t_1, t_2, \ldots, t_{n-1}, t_n = b$.

For each sub-interval $[t_i, t_{i+1}]$, choose

- $s_i \in [t_i, t_{i+1}]$ so that $f(s_i) \leq f(x)$ for all $x \in [t_i, t_{i+1}]$, and
- $l_i \in [t_i, t_{i+1}]$ so that $f(l_i) \geq f(x)$ for all $x \in [t_i, t_{i+1}]$.

The previous theorem guarantees that this is possible.

Now define the *lower Riemann sum* (using n equal subdivisions):

$$\text{Lower sum } = \sum_{i=0}^{n-1} f(s_i)\Delta t,$$

and likewise the *upper Riemann sum* as

$$\text{Upper sum } = \sum_{i=0}^{n-1} f(l_i)\Delta t.$$

To tie this in with what we saw earlier, notice that if f is an increasing function, then lower sums are the same as left sums. But if f is sometimes increasing and sometimes decreasing, left sums might not always give low estimates for $\int_a^b f(x)dx$. But lower Riemann sums are forced to, by their very definition. In other words, for continuous functions, we always have[1]

$$\text{Lower Riemann Sum } \leq \text{exact value of } \int_a^b f(x)dx \leq \text{Upper Riemann Sum}.$$

[1]or we *will*, anyway, after we say what we mean by "exact value of $\int_a^b f(x)dx$".

Next we state a theorem[2]. We saw a similar result to the following in class:

Theorem 0.4. *If f is continuous on $[a, b]$, then*

$$\lim_{n\to\infty} \text{Lower sum} = \lim_{n\to\infty} \text{Upper sum}.$$

We **define** $\int_a^b f(x)dx$ to be their common limit. Then the upper sums close in on it "from above", and the lower ones, "from below".

This definition is an improvement (for reasons we shall see) on the definition in terms of left and right sums, but it is still not perfect. Because it relies on Lemma 0.3, it can break down when f is not a continuous function on a closed interval. There is a way around this, but it requires the same sort of fundamental analysis that the Lemma itself did. Ask me about it if you're curious.

Proof of the FTC

We now assume that f is a continuous function on a closed interval $[a, b]$, and that we have a function F which is an antiderivative of f. We have to show that

$$\int_a^b f(x)dx = F(b) - F(a).$$

Since integrals are defined in terms of upper and lower Riemann sums, that's where we have to start. So fix n for the moment, and suppose we have divided $[a, b]$ into n equal subdivisions as above. Let's focus attention on a representative one: $[t_i, t_{i+1}]$.

By the Mean Value Theorem (applied to F), there is a number $m_i \in [t_i, t_{i+1}]$ so that

$$F'(m_i)(t_{i+1} - t_i) = F(t_{i+1}) - F(t_i),$$

i.e.

$$f(m_i)\Delta t = F(t_{i+1}) - F(t_i).$$

So if s_i and l_i are where the *smallest* and *largest* values of f on $[t_i, t_{i+1}]$ occur, then $f(s_i) \le f(m_i) \le f(l_i)$, and so

$$f(s_i)\Delta t \le f(m_i)\Delta t \le f(l_i)\Delta t,$$

i.e.

$$f(s_i)\Delta t \le F(t_{i+1}) - F(t_i) \le f(l_i)\Delta t.$$

This inequality holds for each i individually, so it holds for the sum of all of the terms, as well. (Stop and think about this.) Hence

$$\sum_{i=0}^{n-1} f(s_i)\Delta t \le \sum_{i=0}^{n-1} (F(t_{i+1}) - F(t_i)) \le \sum_{i=0}^{n-1} f(l_i)\Delta t.$$

Now a wonderful thing occurs: In the sum $\sum_{i=0}^{n-1}(F(t_{i+1}) - F(t_i))$, all the "interior" terms cancel out, and only the ends are left. So

$$\sum_{i=0}^{n-1} (F(t_{i+1} - F(t_i)) = F(t_n) - F(t_0) = F(b) - F(a).$$

Look familiar? So we have

$$\sum_{i=0}^{n-1} f(s_i)\Delta t \le F(b) - F(a) \le \sum_{i=0}^{n-1} f(l_i)\Delta t.$$

[2] I am actually sweeping a lot of technical stuff under the rug by not proving this. So be it.

In words, $F(b) - F(a)$ is trapped between the lower and upper Riemann sum estimates for $\int_a^b f(x)\ dx$.

We did all of this calculation for a fixed, but arbitrary, number of subdivisions n. And if we know that

$$\sum_{i=0}^{n-1} f(s_i)\Delta t \leq F(b) - F(a)$$

for any value of n, we must have

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(s_i)\Delta t \leq F(b) - F(a)$$

as well. It works the same way for upper sums, so

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(s_i)\Delta t \leq F(b) - F(a) \leq \lim_{n\to\infty} \sum_{i=0}^{n-1} f(l_i)\Delta t.$$

But Theorem 0.4 says the two limits are equal, and by *definition*, their common value is $\int_a^b f(x)dx$. It follows that $\int_a^b f(x)dx = F(b) - F(a)$, and so we are done.

Whew!

I realize that this kind of thing is tough going, and it takes a long time for it to illuminate *why* a theorem is really true. So keep in mind the intuition we gained from the examples we studied: If $f = F'$, then value of $f(t)$ measures the change in F per unit time near t. So over a small interval,

(change in F per unit time)$\times$(small time) $\approx$ change in F over a small time interval.

That is,

$$f'(t)\Delta t \approx \Delta F.$$

Adding up the left-hand sides gives us the areas of all the rectangles in the Riemann sum. Adding up all the small changes in F gives the *total* change in F. So the Riemann sum is approximately equal to the total change in F, and taking the limit makes it exact.

THAT is what this formal argument is trying to make precise.