Solutions to the first midterm

April 4, 2009

1. There are actually 8 different correct answers, depending on how you orient the "P", but here I'll show the most obvious, with $T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. I get

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

2a. This is a first-order system (because it's dy/dx, but not d^2y/dx^2 or higher order derivatives).

2b. Equilibrium solutions mean $dy/dx = 0$, which changes $dy/dx + 3y = 6$ into $0 + 3y = 6$, so the solution is $y = 2$.

2c. The equation is separable:

$$\frac{dy}{6 - 3y} = dx \qquad \text{or} \qquad \frac{dy}{y - 2} = -3dx$$

are both ways to separate this.

2d. Yes, the equation is linear.

2e. The general solution (using for example the way we started in 2c) is $y = 2 + Ce^{-3t}$. We determine C by plugging in the initial value $(0, 7)$ and get

$$y = 2 + 5e^{-3t}.$$

3. There are many, many correct solutions to both parts of this question. However, to make this easier to check whether an answer is correct, you'll notice that the two polynomials that I already gave you have the same constant as the coefficient of x^2. This makes it easy for me to check if your answer works.

If you want another polynomial so that the collection DOES span P_2, choose any polynomial ax^2+bx+c with $a \neq c$. For example, you could choose the polynomial 1 (which is also $0x^2+0x+1$).

Notice that if I want to write a polynomial Ax^2+Bx+C as a combination of the three given ones, I can write

$$Ax^2 + Bx + C = (C - A)(1) + \frac{A+B}{2}(x^2 + x + 1) + \frac{A-B}{2}(x^2 - x + 1),$$

which shows that these three polynomials span P_2.

If you want another polynomial so that the collection does NOT span P_2, choose anything in the span of the two given vectors: that is, anything of the form ax^2+bx+a. (So x^2+1 or $3x^2+4x+3$, or so on). You can show it's impossible to get the vector x^2 from any such trio of vectors.

One other way to check whether your answer is correct is to consider the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & a \\ 1 & -1 & b \\ 1 & 1 & c \end{pmatrix}$$

The determinant is 0 if and only if your polynomial is NOT a spanning set. (However remember that the definition of *spanning* doesn't really mention determinants. Rather, *spanning* means that all polynomials can be written as a linear combination of those three special polynomials).

4a.

$$D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3e^{3t} - 42e^{-3t} \\ 6e^{3t} - 21e^{-3t} \end{pmatrix}$$

4b.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^{3t} + 14e^{-3t} \\ 2e^{3t} + 7e^{-3t} \end{pmatrix} = e^{3t}\begin{pmatrix} a+2b \\ c+2d \end{pmatrix} + e^{-3t}\begin{pmatrix} 14a+7b \\ 14c+7d \end{pmatrix}$$

4c. We can find a, b, c, and d by combining parts (a) and (b):

$$\begin{pmatrix} 3 & -42 \\ 6 & -21 \end{pmatrix} = \begin{pmatrix} a+2b & 14a+7b \\ c+2d & 14c+7d \end{pmatrix}$$

This gives us a system of four equations and four unknowns, but it could be worse (it's really just two systems of two equations and two unknowns. Yay!) Solve this any way you can. You should get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ -4 & 5 \end{pmatrix}.$$

5a. We determine the eigenvaules by computing

$$0 = \det\begin{pmatrix} 4-\lambda & 2 \\ \frac{3}{2} & 0-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

so that $\lambda = 1$ or $\lambda = 3$.

5b. For $\lambda = 1$, we solve $\begin{pmatrix} 4 & 2 \\ -\frac{3}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$ yielding $3x = -2y$, so we get $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ (or any scalar multiple of that).

For $\lambda = 3$, we solve $\begin{pmatrix} 4 & 2 \\ -\frac{3}{2} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$ yielding $x = -2y$, so we get $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (or any scalar multiple of that).

5c. The general solution to the differential equation is

$$\begin{pmatrix} x \\ y \end{pmatrix} (t) = Ae^t \begin{pmatrix} 2 \\ -3 \end{pmatrix} + Be^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

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