

solutions to Crannell's practice midterm

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1. It's too hard to do this on the computer. See me if you have questions.

2. We compute the eigenvalue by solving

$$0 = (16 - \lambda)((-11 - \lambda) - (-10)(18)) = 176 - 5\lambda + \lambda^2 + 180,$$

which simplifies nicely to

$$0 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$$

so our eigenvalues are $\lambda = 1$ and $\lambda = 4$.

If we plug these values back into the matrix equation $M\mathbf{v} = \lambda\mathbf{v}$, we get

$$\mathbf{v}_1 = (2, 3)$$

and

$$\mathbf{v}_4 = (5, 6)$$

(any multiples of those vectors would do, too).

3a. (a) is T_2 and (b) is T_1 .

3b. We multiply the matrices in the right order to get

$$\begin{pmatrix} 10 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 10 & 1 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 109 & 25 \\ 25 & 26 \end{pmatrix}$$

so $T_1 \circ T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 109 & 25 \\ 25 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. I won't draw the picture.

4a. We check that

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2,$$

(in particular, the determinant is not 0), so the three vectors are linearly independent. Therefore, we can't say anything about what T does to the third vector unless we get more information.

4b. A not-too-difficult computation tells us that

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so by the linearity of T , we have

$$T \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1/2 \\ 0 \\ 5/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Wow! How nice!

5a. this is homogeneous, linear (but with non-constant coefficients), separable. The easiest way to solve it is to separate it, but you could also use integrating factors. The general solution is $y = Ce^{-x^2}$.

5b. We can guess a particular solution must be a polynomial. Choose a generic polynomial, plug it in and solve for the coefficients to get $y_p = 1$ (that is, a constant function).

5c. Not homogeneous, but yes to all the others.

6. The best way to find the matrix N is to plug the function into the D.E. and then solve for the entries of N . On the left-hand side,

$$D\mathbf{x} = \begin{pmatrix} te^t \\ e^t \end{pmatrix}$$

On the right-hand side, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} te^t - e^t \\ e^t \end{pmatrix} = \begin{pmatrix} (a)te^t + (b-a)e^t \\ (c)te^t + (d-c)e^t \end{pmatrix}.$$

Setting the two sides equal gives us $a = 1$, $b = 1$, $c = 0$, and $d = 1$, so

$$N = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

6b. Because the differential equation is linear, any multiple of a solution is a solution, so

$$47 \begin{pmatrix} te^t - e^t \\ e^t \end{pmatrix}$$

is another solution. Alternatively, you could find the eigenvalue ($\lambda = 1$) and eigenvector $((1, 0))$ of the matrix N , and determine that $(e^t, 0)$ is another solution.

6c. We'll get to this, but we haven't really done enough yet.