

Name _____

Linear Algebra/Differential Equations

Final Exam

December 14, 2006

with Dr. Crannell

problem	points	your score
1	15	_____
2	20	_____
3	16	_____
4	14	_____
5	11	_____
6	14	_____
7	10	_____
Total	100	_____

1. To the right is a fractal generated by the affine transformations

$$T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix},$$

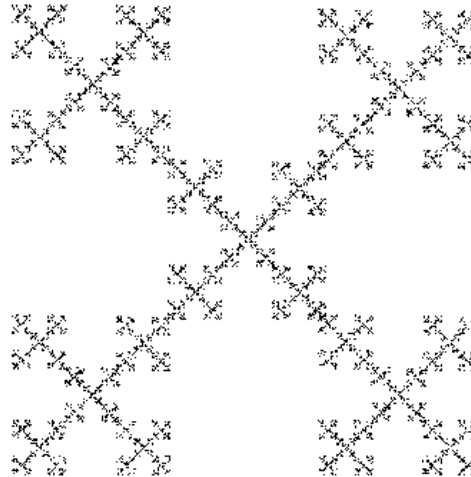
$$T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix},$$

$$T_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix},$$

$$T_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix},$$

and

$$T_5 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$



applied to the unit square I_2 .

a) Label each of the five transformations on the picture in the circles provided.

b) Which of the five affine transformations . . .

. . . has a flip through a horizontal axis (flipping from top to bottom, but not switching left and right)? _____

. . . has a flip through a vertical axis (flipping from left to right, but not switching top and bottom)? _____

. . . rotates the plane counter-clockwise? _____

c) On the fractal, draw a box representing the part of the image which is $T_3(T_2(I^2))$.

d) Compute the Hausdorff dimension of this fractal.

2.

a) Convert the second order the differential equation

$$x'' - 3x' + 2x = 12 \sin(4t) - 14 \cos(4t)$$

into a system of first order differential equations.

b) Determine the general solution to the homogeneous differential equation

$$x'' - 3x' + 2x = 0.$$

c) Determine a particular solution to the differential equation

$$x'' - 3x' + 2x = 12 \sin(4t) - 14 \cos(4t).$$

3. On the next page are
 4 examples of differential equations (marked a–d),
 4 examples of physical descriptions (marked A–D), and
 4 examples of graphs of parametric equations (marked I–IV).

For each of sets of 4, identify which of the examples “does not belong”—meaning it has no counterpart in the other sets of 4. Identify the examples which match up with examples from the other sets.

ANSWERS:	first match	second match	third match	spurious examples
Differential Equations	_____	_____	_____	_____
Physical Descriptions	_____	_____	_____	_____
Graphs	_____	_____	_____	_____

3. (CONTINUED)

DIFFERENTIAL EQUATIONS (The constants a , b , c , d , and e are each assumed to be positive—not negative and not zero—in each equation below).

$$\text{a. } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ 0 \end{pmatrix}$$

$$\text{b. } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ 0 \end{pmatrix}$$

$$\text{c. } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -a & 0 \\ c & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ 0 \end{pmatrix}$$

$$\text{d. } \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} e \\ 0 \end{pmatrix}$$

PHYSICAL DESCRIPTIONS

A. By drinking from a particular mug, a person absorbs a certain amount of lead every day into the blood and tissues. From there, some of the amount of lead in the blood is transferred into the skeleton and some is eliminated through urine, feces, and sweat. A negligible portion (that is, zero) of the lead deposited in the bones leeches back into the bloodstream. We let x stand for the amount of lead in the blood and y stand for the amount of lead in the skeleton.

C. People harvest tuna from the sea; people do not harvest sharks. Sharks and tuna both breed at a rate proportional to their current population. Sharks earn their living (and their reproduction) by eating tuna, but tuna don't kill sharks. We let x stand for the tuna population and y stand for the shark population.

B. At *SquaresRUs* company, an employee's salary increases in two ways: by a fixed dollar amount per year, plus a percentage of the employee's current salary. The amount of funds in the employee's retirement account also increases in two ways: proportionately with his or her current retirement funds, plus a percentage of the employee's current salary. We let x stand for the employee's salary and y for the employee's retirement funds.

D. The Affine Army and the Dihedral Army are at war. Each army conscripts new troops in proportion to its current strength. However, the Affine army has a steady daily influx of additional conscripts which come from their ally, the Boolieans. The war is a paintball war, so nobody dies. We let x stand for the current strength of the Affine Army and y stand for the current strength of the Dihedral army.

GRAPHS OF PARAMETRIC EQUATIONS

I.

II.

III.

IV.

4. Below you will find a list of statements about affine transformations of the plane, *i.e.*, functions of the form $T(\mathbf{x}) = M\mathbf{x} + \mathbf{v}$, where \mathbf{v} is a 2-vector and M is a 2×2 matrix. I will use I^2 to stand for the unit square—the square with opposite corners at $(0,0)$ and $(1,1)$. I will use $T(I^2)$ to stand for the image of the unit square under T .

On this page, M can be **ANY** 2×2 matrix; you might want to compare this with the next page.

In the spaces to the left, indicate whether the given statement is true “always”, “sometimes”, or “never”.

_____ The image of a straight line under T is a straight line.

_____ The area of $T(T(I^2))$ is the square of the area of $T(I^2)$.

_____ Making \mathbf{v} bigger increases the area of $T(I^2)$.

_____ If the area of $T(I^2)$ is less than 1, then $\det(M) < 0$.

_____ The image of a square under T is another square.

_____ The image of a rectangle under T is another rectangle.

_____ The image of a triangle under T is another triangle.

_____ If lines L_1 and L_2 are parallel, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ are parallel.

_____ If lines L_1 and L_2 meet at a 30° angle, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ meet at a 30° angle.

_____ T is one-to-one: that is, if $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

_____ $T \circ T$ is an affine transformation. [Recall that $T \circ T(\mathbf{x}) = T(T(\mathbf{x}))$.]

_____ If T_1 and T_2 are affine transformations, then $T_1 \circ T_2$ is an affine transformation.

_____ If T_1 and T_2 are affine transformations, then $T_1 \circ T_2(\mathbf{x}) = T_2 \circ T_1(\mathbf{x})$.

_____ If T_1 and T_2 are affine transformations, then the area of $T_1 \circ T_2(I^2)$ is equal to the area of $T_2 \circ T_1(I^2)$.

5. Below you will find a list of statements about affine transformations of the plane, *i.e.*, functions of the form $T(\mathbf{x}) = M\mathbf{x} + \mathbf{v}$, where \mathbf{v} is a 2-vector and M is a 2×2 matrix. I will use I^2 to stand for the unit square—the square with opposite corners at $(0,0)$ and $(1,1)$. This square has area 1. I will use $T(I^2)$ to stand for the image of the unit square under T .

On this page, M must have a **non-zero determinant**; you might want to compare this with the previous page.

In the spaces to the left, indicate whether the given statement is true “always”, “sometimes”, or “never”.

_____ The image of a straight line under T is a straight line.

_____ The area of $T(T(I^2))$ is the square of the area of $T(I^2)$.

_____ Making \mathbf{v} bigger increases the area of $T(I^2)$.

_____ If the area of $T(I^2)$ is less than 1, then $\det(M) < 0$.

_____ The image of a square under T is another square.

_____ The image of a rectangle under T is another rectangle.

_____ The image of a triangle under T is another triangle.

_____ If lines L_1 and L_2 are parallel, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ are parallel.

_____ If lines L_1 and L_2 meet at a 30° angle, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ meet at a 30° angle.

_____ T is one-to-one: that is, if $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

_____ $T \circ T$ is an affine transformation. [Recall that $T \circ T(\mathbf{x}) = T(T(\mathbf{x}))$.]

7. Suppose that $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where α , β , γ , and δ are all Real and $\alpha\delta - \gamma\beta < 0$. Consider the following three statements.

- (R) The eigenvalues of M must be Real.
- (C) The eigenvalues of M must be Complex, but not Real.
- (E) The eigenvalues of M could be Real or Complex.

(a) Choose the statement that is true, and give a convincing argument for why it is true. This argument could be either algebraic or geometric.

(b) Choose one statement that is false, and give an example which disproves it.