Name SOLUTIONS________________________________

Linear Algebra/Differential Equations
Final Exam
December 14, 2006

with Dr. Crannell

problem	points	your score
1	15	____________
2	20	____________
3	16	____________
4	14	____________
5	11	____________
6	14	____________
7	10	____________
Total	**100**	____________

1. To the right is a fractal generated by the affine transformations

$$T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1/3 & 0\\0 & 1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/3\\2/3\end{pmatrix},$$

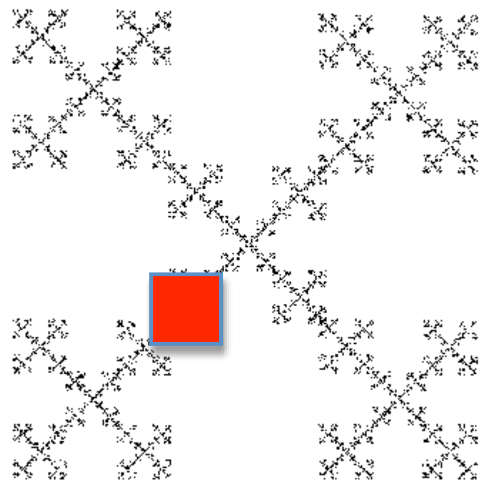
$$T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/3 & 0\\0 & -1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2/3\\1\end{pmatrix},$$

$$T_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & -1/3\\1/3 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2/3\\1/3\end{pmatrix},$$

$$T_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & -1/3\\-1/3 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}1/3\\1/3\end{pmatrix},$$

and

$$T_5\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/3 & 0\\0 & 1/3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}2/3\\0\end{pmatrix}$$



applied to the unit square I_2.

T1: top left   T2: top right   T3: center   T4: bottom left   T5: bottom right

a) Label each of the five transformations on the picture in the circles provided.

b) Which of the five affine transformations . . .

. . . has a flip through a horizontal axis (flipping from top to bottom, but not switching left and right)? _______T2____

. . . has a flip through a vertical axis (flipping from left to right, but not switching top and bottom)? _____T1____

. . . rotates the plane counter-clockwise? ____T3____

c) On the fractal, draw a box representing the part of the image which is $T_3\left(T_2\left(I^2\right)\right)$.

Where the red box is

d) Compute the Hausdorff dimension of this fractal.

Log(5)/log(3)

2.

a) Convert the second order the differential equation

$$x'' - 3x' + 2x = 12\sin(4t) - 14\cos(4t)$$

into a system of first order differential equations.

x' = y
y' = 3y – 2x + 12 sin(4t) – 14cos(4t)

b) Determine the general solution to the homogeneous differential equation

$$x'' - 3x' + 2x = 0.$$

The characteristic equation is $r^2 - 3r + 2 = 0$ with roots r = 1 and r =2, so the general solution is

$x(t) = A e^t + B e^{2t}$

c) Determine a particular solution to the differential equation

$$x'' - 3x' + 2x = 12\sin(4t) - 14\cos(4t).$$

We guess x = a sin(4t) + b cos(4t) and plug this in to the equation. This gives us a sin equation and a cosine equation:

(-16a + 12b + 2a = 12) and
(-16b – 12a + 2b = -14),

which we can solve to get the fortunately nice answer a = 0 and b = 1, so

$x_p = \cos(4t)$.

3. On the next page are

   4 examples of differential equations (marked a–d),
   4 examples of physical descriptions (marked A–D), and
   4 examples of graphs of parametric equations (marked I–IV).

For each of sets of 4, identify which of the examples "does not belong"—meaning it has no counterpart in the other sets of 4. Identify the examples which match up with examples from the other sets.

ANSWERS:	first match	second match	third match	spurious examples
Differential Equations	c__________	a__________	d__________	b__________
Physical Descriptions	A__________	B__________	C__________	D__________
Graphs	___________	___________	___________	___________

3. (CONTINUED)

DIFFERENTIAL EQUATIONS (The constants a, b, c, d, and e are each assumed to be positive—not negative and not zero—in each equation below).

a. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ 0 \end{pmatrix}$

b. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ 0 \end{pmatrix}$

c. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -a & 0 \\ c & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ 0 \end{pmatrix}$

d. $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & -b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} e \\ 0 \end{pmatrix}$

PHYSICAL DESCRIPTIONS

A. By drinking from a particular mug, a person absorbs a certain amount of lead every day into the blood and tissues. From there, some of the amount of lead in the blood is transferred into the skeleton and some is eliminated through urine, feces, and sweat. A negligible portion (that is, zero) of the lead deposited in the bones leeches back into the bloodstream. We let x stand for the amount of lead in the blood and y stand for the amount of lead in the skeleton.

C. People harvest tuna from the sea; people do not harvest sharks. Sharks and tuna both breed at a rate proportional to their current population. Sharks earn their living (and their reproduction) by eating tuna, but tuna don't kill sharks. We let x stand for the tuna population and y stand for the shark population.

B. At *SquaresRUs* company, an employee's salary increases in two ways: by a fixed dollar amount per year, plus a percentage of the employee's current salary. The amount of funds in the employee's retirement account also increases in two ways: proportionately with his or her current retirement funds, plus a percentage of the employee's current salary. We let x stand for the employee's salary and y for the employee's retirement funds.

D. The Affine Army and the Dihedral Army are at war. Each army conscripts new troops in proportion to its current strength. However, the Affine army has a steady daily influx of additional conscripts which come from their ally, the Boolieans. The war is a paintball war, so nobody dies. We let x stand for the current strength of the Affine Army and y stand for the current strength of the Dihedral army.

GRAPHS OF PARAMETRIC EQUATIONS

I.

II.

III.

IV.

4. Below you will find a list of statements about affine transformations of the plane, *i.e.*, functions of the form $T(\mathbf{x}) = M\mathbf{x} + \mathbf{v}$, where $\mathbf{v}$ is a 2-vector and M is a 2x2 matrix. I will use I^2 to stand for the unit square—the square with opposite corners at (0,0) and (1,1). I will use $T(I^2)$ to stand for the image of the unit square under T.

On this page, *M* can be **ANY** 2x2 matrix; you might want to compare this with the next page.

In the spaces to the left, indicate whether the given statement is true "always", "sometimes", or "never".

S (or a point)_____The image of a straight line under T is a straight line.

A_______________The area of $T(T(I^2))$ is the square of the area of $T(I^2)$.

N_______________Making **v** bigger increases the area of $T(I^2)$.

S_______________If the area of $T(I^2)$ is less than 1, then $\det(M)<0$.

S _______________The image of a square under T is another square.

S_______________The image of a rectangle under T is another rectangle.

S (or a line or a point) The image of a triangle under T is another triangle.

A_______________If lines L_1 and L_2 are parallel, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ are parallel.

S_______________If lines L_1 and L_2 meet at a 30° angle, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ meet at a 30° angle.

S_______________T is one-to-one: that is, if $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$.

A_______________$T \circ T$ is an affine transformation. [Recall that $T \circ T(x) = T(T(x))$.]

A_______________If T_1 and T_2 are affine transformations, then $T_1 \circ T_2$ is an affine transformation.

S_______________If T_1 and T_2 are affine transformations, then $T_1 \circ T_2(\mathbf{x}) = T_2 \circ T_1(\mathbf{x})$.

A_______________If T_1 and T_2 are affine transformations, then the area of $T_1 \circ T_2(I^2)$ is equal to the area of $T_2 \circ T_1(I^2)$.

5. Below you will find a list of statements about affine transformations of the plane, *i.e.*, functions of the form T(**x**) = M**x** + **v**, where **v** is a 2-vector and M is a 2x2 matrix. I will use I^2 to stand for the unit square—the square with opposite corners at (0,0) and (1,1). This square has area 1. I will use $T(I^2)$ to stand for the image of the unit square under T.

On this page, *M* must have a **non-zero determinant**; you might want to compare this with the previous page.

In the spaces to the left, indicate whether the given statement is true "always", "sometimes", or "never".

A______________The image of a straight line under T is a straight line.

A______________The area of $T(T(I^2))$ is the square of the area of $T(I^2)$.

N______________Making **v** bigger increases the area of $T(I^2)$.

S______________If the area of $T(I^2)$ is less than 1, then $\det(M)<0$.

S______________The image of a square under T is another square.

S______________The image of a rectangle under T is another rectangle.

A______________The image of a triangle under T is another triangle.

A______________If lines L_1 and L_2 are parallel, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ are parallel.

S______________If lines L_1 and L_2 meet at a 30° angle, and if $T(L_1)$ and $T(L_2)$ are lines, then $T(L_1)$ and $T(L_2)$ meet at a 30° angle.

A______________T is one-to-one: that is, if T(**x**) = T(**y**), then **x** = **y**.

A______________T∘T is an affine transformation. [Recall that T∘T(x) = T(T(x)).]

6.

a) Which of the following (if any) are vector subspaces of $\mathbf{R}^2$?

$$V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| a = 3 + b \right\} \qquad U = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| a = 3b \right\} \qquad W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| a \geq b \right\}$$

answer:_______U_____

V does not contain (0,0), and W is not closed under scalar multiplication (i.e., multiply by -1)

b) Which of the following (if any) are linear transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$?

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x + y \end{pmatrix}$$

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3 + y \end{pmatrix}$$

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3y \end{pmatrix}$$

$$J\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 3 \end{pmatrix}$$

answer:_______F, H___

G and J don't send (0,0) to (0,0), and they don't preserve vector addition

7. Suppose that $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where α, β, γ, and δ are all Real and $\alpha\delta - \gamma\beta < 0$. Consider the following three statements.

(**R**) The eigenvalues of *M* must be Real.
(**C**) The eigenvalues of *M* must be Complex, but not Real.
(**E**) The eigenvalues of *M* could be Real or Complex.

(a) Choose the statement that is true, and give a convincing argument for why it is true. This argument could be either algebraic or geometric.

R: The determinant is negative, so the matrix flips the plane. The axis about which the plane flips stays in the same direction – that is, it's an eigen direction, so that must have a positive real eigenvalue.

(b) Choose one statement that is false, and give an example which disproves it.

C: Let M =

-2	0
0	4.

This has determinant -8, and real eigenvalues (-2 and 4).